Loday algebroids and their supergeometric interpretation

Janusz Grabowski*, David Khudaverdyan, and Norbert Poncin†

Abstract

A concept of *Loday algebroid* (and its pure algebraic version – *Loday pseudoalgebra*) is proposed and discussed in comparison with other similar structures present in the literature. The structure of a Loday pseudoalgebra and its natural reduction to a Lie pseudoalgebra is studied. Further, Loday algebroids are interpreted as homological vector fields on a 'supercommutative manifold' and the corresponding Cartan calculus is introduced. Several examples, including Courant algebroids, Grassmann-Dorfman and twisted Courant-Dorfman brackets, as well as algebroids induced by Nambu-Poisson structures, are given.

MSC 2000: 53D17, 58A50, 17A32, 17B66

Keywords: Algebroid, pseudoalgebra, Loday algebra, Courant bracket, supercommutative manifold, homological vector field, Cartan calculus

1 Introduction

The concept of *Dirac structure*, proposed by Dorfman [4] in the Hamiltonian framework of integrable evolution equations and defined in [3] as an isotropic subbundle of the Whitney sum $\mathcal{T}M = \mathsf{T}M \oplus_M \mathsf{T}^*M$ of the tangent and the cotangent bundles and satisfying some additional conditions, provides a geometric setting for Dirac's theory of constrained mechanical systems. To formulate the integrability condition defining the Dirac structure, Courant [3] introduced a natural skew-symmetric bracket operation on sections of $\mathcal{T}M$. The Courant bracket does not satisfy the Leibniz rule with respect to multiplication by functions nor the Jacobi identity. These defects disappear upon restriction to a Dirac subbundle because of the isotropy condition. Particular cases of Dirac structures are graphs of closed 2-forms and Poisson bivector fields on the manifold M.

The nature of the Courant bracket itself remained unclear until several years later when it was observed by Liu, Weinstein and Xu [32] that $\mathcal{T}M$ endowed with the Courant bracket plays the role of a 'double' object, in the sense of Drinfeld [5], for a pair of Lie algebroids (see [34]) over M. Let us recall that, in complete analogy with Drinfeld's Lie bialgebras, in the category of Lie algebroids there also exist 'bi-objects', Lie bialgebroids, introduced by Mackenzie and Xu [35] as linearizations of Poisson groupoids. On the other hand, every Lie bialgebra has a double which is a Lie algebra. This is not so for general Lie bialgebroids. Instead, Liu, Weinstein and Xu [32] showed that the double of a Lie bialgebroid is a more complicated structure they call a *Courant algebroid*, $\mathcal{T}M$ with the Courant bracket being a special case.

*The research of J. Grabowski was supported by the Polish Ministry of Science and Higher Education under the grant N N201 416839.

†The research of N. Poncin was supported by Grant GeoAlgPhys 2011-2013 awarded by the University of Luxembourg.

There is also another way of viewing Courant algebroids as a generalization of Lie algebroids. This requires a change in the definition of the Courant bracket and considering an analog of the non-antisymmetric Dorfman bracket [4], so that the traditional Courant bracket becomes the skew-symmetrization of the new one [43]. This change replaces one of the defects with another one: a version of the Jacobi identity is satisfied, while the bracket is no longer skew-symmetric. Such algebraic structures have been introduced by Loday [30] under the name *Leibniz algebras*, but they are nowadays also often called *Loday algebras*. Loday algebras, like their skew-symmetric counterparts – Lie algebras – determine certain cohomological complexes, defined on tensor algebras instead of Grassmann algebras.

Since Loday brackets, like the Courant-Dorfman bracket, appear naturally in Geometry and Physics in the form of 'algebroid brackets', i.e. brackets on sections of vector bundles, there were several attempts to formalize the concept of *Loday* (or *Leibniz*) *algebroid* (see e.g. [1, 11, 13, 21, 19, 20, 27, 36, 45, 48]). We prefer the terminology *Loday algebroid* to distinguish them from other *general algebroid* brackets with both anchors (see [17]), called sometimes *Leibniz algebroids* or *Leibniz brackets* and used recently in Physics, for instance, in the context of nonholonomic constraints [8, 9, 12, 38].

The concepts of Loday algebroid we found in the literature do not seem to be exactly appropriate. The notion in [11], which assumes the existence of both anchor maps, is too strong and admits no real new examples, except for Lie algebroids and bundles of Loday algebras. The concept introduced in [45] requires a pseudo-Riemannian metric on the bundle, so it is too strong as well and does not reduce to a Loday algebra when we consider a bundle over a single point, while the other concepts [21, 19, 20, 27, 36, 48], assuming only the existence of a left anchor, do not put any differentiability requirements for the first variable, so that they are not geometric and too weak (see Example 4.3). Only in [1] one considers some Leibniz algebroids with local brackets.

The aim of this work is to propose a modified concept of Loday algebroid in terms of an operation on sections of a vector bundle, as well as in terms of a homological vector field of a supercommutative manifold. We put some minimal requirements that a proper concept of Loday algebroid should satisfy. Namely, the definition of Loday algebroid, understood as a certain operation on sections of a vector bundle E,

- should reduce to the definition of Loday algebra in the case when E is just a vector space;
- should contain the Courant-Dorfman bracket as a particular example;
- should be as close to the definition of Lie algebroid as possible.

We propose a definition satisfying all these requirements and including all main known examples of Loday brackets with geometric origins. Moreover, we can interpret our Loday algebroid structures as homological vector fields on a supercommutative manifold; this opens, like in the case of Lie algebroids, new horizons for a geometric understanding of these objects and of their possible 'higher generalizations'.

The paper is organized as follows. We first recall – in Section 2 – needed results on differential operators and derivative endomorphisms. In Section 3 we investigate, under the name of pseudoalgebras, algebraic counterparts of algebroids requiring varying differentiability properties for the two entries of the bracket. The results of Section 4 show that we should relax our traditional understanding of the right anchor map. A concept of Loday algebroid satisfying all the above requirements is proposed in Definition 4.7 and further detailed in Theorem 4.8. In Section 5 we describe a number of new Loday algebroids containing main canonical examples of Loday brackets on sections of a vector bundle. A natural reduction a Loday pseudoalgebra to a Lie pseudoalgebra is studied in Section 6. For the standard Courant bracket it corresponds

to its reduction to the Lie bracket of vector fields. We then define Loday algebroid cohomology, Section 7, and interpret in Section 8 our Loday algebroid structures in terms of homological vector fields of the graded ringed space given by the shuffle multiplication of multidifferential operators, see Theorem 8.6. We introduce also the corresponding Cartan calculus.

2 Differential operators and derivative endomorphisms

All geometric objects, like manifolds, bundles, maps, sections, etc. will be smooth throughout this paper.

Definition 2.1. A *Lie algebroid* structure on a vector bundle $\tau : E \to M$ is a Lie algebra bracket $[\cdot,\cdot]$ on the real vector space $\mathcal{E} = \mathsf{Sec}(E)$ of sections of E which satisfies the following compatibility condition related to the $\mathcal{A} = C^\infty(M)$-module structure in $\mathcal{E}$:

$$\forall\ X, Y \in \mathcal{E}\ \forall f \in \mathcal{A} \quad [X, fY] - f[X, Y] = \rho(X)(f)Y\,, \tag{1}$$

for some vector bundle morphism $\rho : E \to \mathsf{T}M$ covering the identity on M and called the *anchor map*. Here, $\rho(X) = \rho \circ X$ is the vector field on M associated *via* ρ with the section X.

Note that the bundle morphism ρ is uniquely determined by the bracket of the Lie algebroid. What differs a general Lie algebroid bracket from just a Lie module bracket on the $C^\infty(M)$-module $\mathsf{Sec}(E)$ of sections of E is the fact that it is not $\mathcal{A}$-bilinear but a certain first-order bidifferential operator: the adjoint operator $\mathrm{ad}_X = [X, \cdot]$ is a *derivative endomorphism*, i.e., the *Leibniz rule*

$$\mathrm{ad}_X(fY) = f\mathrm{ad}_X(Y) + \widehat{X}(f)Y \tag{2}$$

is satisfied for each $Y \in \mathcal{E}$ and $f \in \mathcal{A}$, where $\widehat{X} = \rho(X)$ is the vector field on M assigned to X, the *anchor* of X. Moreover, the assignment $X \mapsto \widehat{X}$ is a differential operator of order 0, as it comes from a bundle map $\rho : E \mapsto \mathsf{T}M$.

Derivative endomorphisms (also called *quasi-derivations*), like differential operators in general, can be defined for any module $\mathcal{E}$ over an associative commutative ring $\mathcal{A}$. Also an extension to superalgebras is straightforward. These natural ideas go back to Grothendieck and Vinogradov [47]. On the module $\mathcal{E}$ we have namely a distinguished family $\mathcal{A}_\mathcal{E} = \{f_\mathcal{E} : f \in \mathcal{A}\}$ of linear operators provided by the module structure: $f_\mathcal{E}(Y) = fY$.

Definition 2.2. Let $\mathcal{E}_i$, $i = 1, 2$, be modules over the same ring $\mathcal{A}$. We say that an additive operator $D : \mathcal{E}_1 \to \mathcal{E}_2$ is a *differential operator of order 0*, if it intertwines $f_{\mathcal{E}_1}$ with $f_{\mathcal{E}_2}$, i.e.

$$\delta(f)(D) := D \circ f_{\mathcal{E}_1} - f_{\mathcal{E}_2} \circ D\,, \tag{3}$$

vanishes for all $f \in \mathcal{A}$. Inductively, we say that D is a *differential operator of order* $\le k + 1$, if the commutators (3) are differential operators of order $\le k$. In other words, D is a differential operator of order $\le k$ if and only if

$$\forall\ f_1, \dots, f_{k+1} \in \mathcal{A} \quad \delta(f_1)\delta(f_2)\cdots\delta(f_{k+1})(D) = 0\,. \tag{4}$$

The corresponding set of differential operators of order $\le k$ will be denoted by $\mathcal{D}_k(\mathcal{E}_1; \mathcal{E}_2)$ (shortly, $\mathcal{D}_k(\mathcal{E})$, if $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$) and the set of differential operators of arbitrary order (filtered by $(\mathcal{D}_k(\mathcal{E}_1; \mathcal{E}_2))_{k=0}^\infty$) by $\mathcal{D}(\mathcal{E}_1; \mathcal{E}_2)$ (resp., $\mathcal{D}(\mathcal{E})$). We will say that D *is of order* k if it is of order $\le k$ and not of order $\le k - 1$.

In particular, $\mathcal{D}_0(\mathcal{E}_1; \mathcal{E}_2) = \mathsf{Hom}_\mathcal{A}(\mathcal{E}_1; \mathcal{E}_2)$ is made up by module homomorphisms. Note that in the case when $\mathcal{E}_i = \mathsf{Sec}(E_i)$ is the module of sections of a vector bundle E_i, $i = 1, 2$, the

concept of differential operators defined above coincides with the standard understanding. As this will be our standard geometric model, to reduce algebraic complexity we will assume that $\mathcal{A}$ is an associative commutative algebra with unity 1 over a field $\mathbb{K}$ of characteristic 0 and all the $\mathcal{A}$-modules are faithful. In this case, $\mathcal{D}(\mathcal{E}_1;\mathcal{E}_2)$ is a (canonically filtered) vector space over $\mathbb{K}$ and, since we work with fields of characteristic 0, condition (4) is equivalent to a simpler condition (see [10])

$$\forall\, f\in\mathcal{A}\quad \delta(f)^{k+1}(D)=0\,. \tag{5}$$

If $\mathcal{E}_1=\mathcal{E}_2=\mathcal{E}$, then $\delta(f)(D)=[D,f_{\mathcal{E}}]_c$, where $[\cdot,\cdot]_c$ is the commutator bracket, and elements of $\mathcal{A}_{\mathcal{E}}$ are particular 0-order operators. Therefore, we can canonically identify $\mathcal{A}$ with the subspace $\mathcal{A}_{\mathcal{E}}$ in $\mathcal{D}_0(\mathcal{E})$ and use it to distinguish a particular set of first-order differential operators on $\mathcal{E}$ as follows.

Definition 2.3. *Derivative endomorphisms* (or *quasi-derivations*) $D:\mathcal{E}\to\mathcal{E}$ are particular first-order differential operators distinguished by the condition

$$\forall\, f\in\mathcal{A}\quad \exists\,\widehat{f}\in\mathcal{A}\quad [D,f_{\mathcal{E}}]_c=\widehat{f}_{\mathcal{E}}\,. \tag{6}$$

Since the commutator bracket satisfies the Jacobi identity, one can immediately conclude that $\widehat{f}_{\mathcal{E}}=\widehat{D}(f)_{\mathcal{E}}$ which holds for some derivation $\widehat{D}\in\mathsf{Der}(\mathcal{A})$ and an arbitrary $f\in\mathcal{A}$ [11]. Derivative endomorphisms form a submodule $\mathsf{Der}(\mathcal{E})$ in the $\mathcal{A}$-module $\mathsf{End}_{\mathbb{K}}(\mathcal{E})$ of $\mathbb{K}$-linear endomorphisms of $\mathcal{E}$ which is simultaneously a Lie subalgebra over $\mathbb{K}$ with respect to the commutator bracket. The linear map,

$$\mathsf{Der}(\mathcal{E})\ni D\mapsto\widehat{D}\in\mathsf{Der}(\mathcal{A})\,,$$

called the *universal anchor map*, is a differential operator of order 0, $\widehat{fD}=f\widehat{D}$. The Jacobi identity for the commutator bracket easily implies (see [11, Theorem 2])

$$\widehat{[D_1,D_2]_c}=[\widehat{D_1},\widehat{D_2}]_c\,. \tag{7}$$

It is worth remarking (see [11]) that also $\mathcal{D}(\mathcal{E})$ is a Lie subalgebra in $\mathsf{End}_{\mathbb{K}}(\mathcal{E})$, as

$$[\mathcal{D}_k(\mathcal{E}),\mathcal{D}_l(\mathcal{E})]_c\subset\mathcal{D}_{k+l-1}(\mathcal{E})\,, \tag{8}$$

and an associative subalgebra, as

$$\mathcal{D}_k(\mathcal{E})\circ\mathcal{D}_l(\mathcal{E})\subset\mathcal{D}_{k+l}(\mathcal{E})\,, \tag{9}$$

that makes $\mathcal{D}(\mathcal{E})$ into a canonical example of a *quantum Poisson algebra* in the terminology of [16].

It was pointed out in [26] that the concept of derivative endomorphism can be traced back to N. Jacobson [22, 23] as a special case of his *pseudo-linear endomorphism*. It has appeared also in [37] under the name *module derivation* and was used to define linear connections in the algebraic setting. In the geometric setting of Lie algebroids it has been studied in [34] under the name *covariant differential operator*. For more detailed history and recent development we refer to [26].

Algebraic operations in differential geometry have usually a local character in order to be treatable with geometric methods. On the pure algebraic level we should work with differential (or multidifferential) operations, as tells us the celebrated Peetre Theorem [39, 40]. The algebraic concept of a multidifferential operator is obvious. For a $\mathbb{K}$-multilinear operator $D:\mathcal{E}_1\times\cdots\times\mathcal{E}_p\to\mathcal{E}$ and each $i=1,\dots,p$, we say that D is a *differential operator of order $\le k$ with respect to the ith variable*, if, for all $y_j\in\mathcal{E}_j$, $j\ne i$,

$$D(y_1,\dots,y_{i-1},\,\cdot\,,y_{i+1},\dots,y_p):\mathcal{E}_i\to\mathcal{E}$$

is a differential operator of order $\leq k$. In other words,

$$\forall\, f \in \mathcal{A} \quad \delta_i(f)^{k+1}(D) = 0\,, \tag{10}$$

where

$$\delta_i(f)D(y_1, \dots, y_p) = D(y_1, \dots, f y_i, \dots, y_p) - fD(y_1, \dots, y_p)\,. \tag{11}$$

Note that the operations $\delta_i(f)$ and $\delta_j(g)$ commute. We say that the operator D *is a multidifferential operator of order* $\leq n$, if it is of order $\leq n$ with respect to each variable separately. This means that, fixing any $p-1$ arguments, we get a differential operator of order $\leq n$. A similar, but stronger, definition is the following

Definition 2.4. We say that a multilinear operator $D : \mathcal{E}_1 \times \cdots \times \mathcal{E}_p \to \mathcal{E}$ is a *multidifferential operator of total order* $\leq k$, if

$$\forall\, f_1, \dots, f_{k+1} \in \mathcal{A}\ \forall\, i_1, \dots, i_{k+1} = 1, \dots, p \quad \left[\delta_{i_1}(f_1)\delta_{i_2}(f_2) \cdots \delta_{i_{k+1}}(f_{k+1})(D) = 0\right]\,. \tag{12}$$

Of course, a multidifferential operator of total order $\leq k$ is a multidifferential operator of order $\leq k$. It is also easy to see that a p-linear differential operator of order $\leq k$ is a multidifferential operator of total order $\leq pk$. In particular, the Lie bracket of vector fields (in fact, any Lie algebroid bracket) is a bilinear differential operator of total order ≤ 1.

3 Pseudoalgebras

Let us start this section with recalling that Loday, while studying relations between Hochschild and cyclic homology in the search for obstructions to the periodicity of algebraic K-theory, discovered that one can skip the skew-symmetry assumption in the definition of Lie algebra, still having a possibility to define an appropriate (co)homology (see [29, 31] and [30, Chapter 10.6]). His Jacobi identity for such structures was formally the same as the classical Jacobi identity in the form

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]. \tag{13}$$

This time, however, this is no longer equivalent to

$$[[x, y], z] = [[x, z], y] + [x, [y, z]], \tag{14}$$

nor to

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \tag{15}$$

since we have no skew-symmetry. Loday called such structures *Leibniz algebras*, but to avoid collision with another concept of *Leibniz brackets* in the literature, we shall call them *Loday algebras*. This is in accordance with the terminology of [K-S], where analogous structures in the graded case are defined. Note that the identities (13) and (14) have an advantage over the identity (15) obtained by cyclic permutations, since they describe the algebraic facts that the left-regular (resp., right-regular) actions are left (resp., right) derivations. This was the reason to name the structure 'Leibniz algebra'.

Of course, there is no particular reason not to define Loday algebras by means of (14) instead of (13) (and in fact, it was the original definition by Loday), but this is not a substantial difference, as both categories are equivalent via transposition of arguments. We will use the form (13) of the Jacobi identity.

Our aim is to find a proper generalization of the concept of Loday algebra in a way similar to that in which Lie algebroids generalize Lie algebras. If one thinks about a generalization of a concept of Lie algebroid as operations on sections of a vector bundle including operations

(brackets) which are non-antisymmetric or which do not satisfy the Jacobi identity, and are not just $\mathcal{A}$-bilinear, then it is reasonable, on one hand, to assume differentiability properties of the bracket as close to the corresponding properties of Lie algebroids as possible and, on the other hand, including all known natural examples of such brackets. This is not an easy task, since, as we will see soon, some natural possibilities provide only few new examples.

To present a list of these possibilities, we propose the following definitions serving in the pure algebraic setting.

Definition 3.1. Let $\mathcal{E}$ be a faithful module over an associative commutative algebra $\mathcal{A}$ over a field $\mathbb{K}$ of characteristic 0. A a $\mathbb{K}$-bilinear bracket $B = [\cdot,\cdot] : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ on the module $\mathcal{E}$

1. is called a *Kirillov pseudoalgebra bracket*, if B is a bidifferential operator;
2. is called a *weak pseudoalgebra bracket*, if B is a bidifferential operator of degree ≤ 1;
3. is called a *quasi pseudoalgebra bracket*, if B is a bidifferential operator of total degree ≤ 1;
4. is called a *pseudoalgebra bracket*, if B is a bidifferential operator of total degree ≤ 1 and the *adjoint map* $\mathrm{ad}_X = [X,\cdot] : \mathcal{E} \to \mathcal{E}$ is a derivative endomorphism for each $X \in \mathcal{E}$;
5. is called a *QD-pseudoalgebra bracket*, if the *adjoint maps* $\mathrm{ad}_X, \mathrm{ad}^r_X : \mathcal{E} \to \mathcal{E}$,
$$\mathrm{ad}_X = [X,\cdot]\,, \quad \mathrm{ad}^r_X = [\cdot, X] \quad (X \in \mathcal{E})\,, \tag{16}$$
associated with B are derivative endomorphisms (quasi-derivations);
6. is called a *strong pseudoalgebra bracket*, if B is a bidifferential operator of total degree ≤ 1 and the *adjoint maps* $\mathrm{ad}_X, \mathrm{ad}^r_X : \mathcal{E} \to \mathcal{E}$,
$$\mathrm{ad}_X = [X,\cdot]\,, \quad \mathrm{ad}^r_X = [\cdot, X] \quad (X \in \mathcal{E})\,, \tag{17}$$
are derivative endomorphisms.

We call the module $\mathcal{E}$ equipped with such a bracket, respectively, a *Kirillov pseudoalgebra*, *weak pseudoalgebra* etc. If the bracket is symmetric (skew-symmetric), we speak about Kirillov, weak, etc., *symmetric* (*skew*) *pseudoalgebras*. If the bracket satisfies the Jacobi identity (13), we speak about local, weak, etc., *Loday pseudoalgebras*, and if the bracket is a Lie algebra bracket, we speak about local, weak, etc., *Lie pseudoalgebras*. If $\mathcal{E}$ is the $\mathcal{A} = C^\infty(M)$ module of sections of a vector bundle $\tau : E \to M$, we refer to the above pseudoalgebra structures as to *algebroids*.

Theorem 3.2. *If* $[\cdot,\cdot]$ *is a pseudoalgebra bracket, then the map*
$$\rho : \mathcal{E} \to \mathsf{Der}(\mathcal{A})\,, \quad \rho(X) = \widehat{\mathrm{ad}_X}\,,$$
called the anchor map, *is* $\mathcal{A}$*-linear,* $\rho(fX) = f\rho(X)$*, and*
$$[X, fY] = f[X,Y] + \rho(X)(f)Y \tag{18}$$
for all $X, Y \in \mathcal{E}$*,* $f \in \mathcal{A}$*. Moreover, if* $[\cdot,\cdot]$ *satisfies additionally the Jacobi identity, i.e., we deal with a Loday pseudoalgebra, then the anchor map is a homomorphism into the commutator bracket,*
$$\rho\left([X,Y]\right) = [\rho(X), \rho(Y)]_c\,. \tag{19}$$

Proof. Since the bracket B is a bidifferential operator of total degree ≤ 1, we have $\delta_1(f)\delta_2(g)B = 0$ for all $f, g \in \mathcal{A}$. On the other hand, as easily seen,

$$(\delta_1(f)\delta_2(g)B)(X,Y) = (\rho(fX) - f\rho(X))(g)Y\,, \tag{20}$$

and the module is faithful, it follows $\rho(fX) = f\rho(X)$. The identity (19) is a direct implication of the Jacobi identity combined with (18).

□

Theorem 3.3. *If $[\cdot,\cdot]$ is a QD-pseudoalgebra bracket, then it is a weak pseudoalgebra bracket and admits two anchor maps*

$$\rho, \rho^r : \mathcal{E} \to \mathsf{Der}(\mathcal{A})\,, \quad \rho(X) = \widehat{\mathrm{ad}_X}\,,\ \rho^r = -\widehat{\mathrm{ad}^r}\,,$$

for which we have

$$[X, fY] = f[X,Y] + \rho(X)(f)Y\,, \quad [fX, Y] = f[X,Y] - \rho^r(X)(f)Y\,, \tag{21}$$

for all $X, Y \in \mathcal{E}$, $f \in \mathcal{A}$. If the bracket is skew-symmetric, then both anchors coincide, and if the bracket is a strong QD-pseudoalgebra bracket, they are $\mathcal{A}$-linear. Moreover, if $[\cdot,\cdot]$ satisfies additionally the Jacobi identity, i.e., we deal with a Loday QD-pseudoalgebra, then, for all $X, Y \in \mathcal{E}$,

$$\rho([X,Y]) = [\rho(X), \rho(Y)]_c\,. \tag{22}$$

Proof. Similarly as above,

$$(\delta_2(f)\delta_2(g)B)(X,Y) = \rho(X)(g)fY - f\rho(X)(g)Y = 0\,,$$

so B is a first-order differential operator with respect to the second argument. The same can be done for the first argument.

Next, as for any QD-pseudoalgebra bracket B we have, analogously to (20),

$$(\delta_1(f)\delta_2(g)B)(X,Y) = (\rho(fX) - f\rho(X))(g)Y = (\rho^r(gY) - g\rho^r(Y))(f)X\,, \tag{23}$$

both anchor maps are $\mathcal{A}$-linear if and only if D is of total order ≤ 1. The rest follows analogously to the previous theorem.

□

The next observation is that quasi pseudoalgebra structures on an $\mathcal{A}$-module $\mathcal{E}$ have certain analogs of anchor maps, namely $\mathcal{A}$-module homomorphisms $b = b^r, b^l : \mathcal{E} \to \mathsf{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \mathsf{End}(\mathcal{E})$. For every $X \in \mathcal{E}$ we will view $b(X)$ as an $\mathcal{A}$-module homomorphism $b(X) : \Omega^1 \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$, where Ω^1 is the $\mathcal{A}$-submodule of $\mathsf{Hom}_{\mathcal{A}}(\mathsf{Der}(\mathcal{A}); \mathcal{A})$ generated by $\mathrm{d}\mathcal{A} = \{\mathrm{d}f : f \in \mathcal{A}\}$ and $\langle \mathrm{d}f, D\rangle = D(f)$. Elements of $\mathsf{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \mathsf{End}(\mathcal{E})$ act on elements of $\Omega^1 \otimes_{\mathcal{A}} \mathcal{E}$ in the obvious way: $(V \otimes \Phi)(\omega \otimes X) = \langle V, \omega\rangle \Phi(X)$.

Theorem 3.4. *A $\mathbb{K}$-bilinear bracket $B = [\cdot,\cdot]$ on an $\mathcal{A}$-module $\mathcal{E}$ defines a quasi pseudoalgebra structure if and only if there are $\mathcal{A}$-module homomorphisms*

$$b^r, b^l : \mathcal{E} \to \mathsf{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \mathsf{End}(\mathcal{E})\,, \tag{24}$$

called generalized anchor maps, right and left, *such that, for all $X, Y \in \mathcal{E}$ and all $f \in \mathcal{A}$,*

$$[X, fY] = f[X,Y] + b^l(X)(\mathrm{d}f \otimes Y)\,, \quad [fX, Y] = f[X,Y] - b^r(Y)(\mathrm{d}f \otimes X)\,. \tag{25}$$

The generalized anchor maps are actual anchor maps if they take values in $\mathsf{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \{\mathrm{Id}_{\mathcal{E}}\}$.

Proof. Assume first that the bracket B is a bidifferential operator of total degree ≤ 1 and define a three-linear map of vector spaces $A:\mathcal{E}\times\mathcal{A}\times\mathcal{E}\to\mathcal{E}$ by

$$A(X,g,Y)=(\delta_2(g)B)(X,Y)=[X,gY]-g[X,Y]\,.$$

It is easy to see that A is $\mathcal{A}$-linear with respect to the first and the third argument, and a derivation with respect to the second. Indeed, as

$$(\delta_1(f)\delta_2(g)B)(X,Y)=A(fX,g,Y)-fA(X,g,Y)=0\,,$$

we get $\mathcal{A}$-linearity with respect to the first argument. Similarly, from $\delta_2(f)\delta_2(g)B=0$, we get the same conclusion for the third argument. We have also

$$\begin{aligned} A(X,fg,Y) &= [X,fgY]-fg[X,Y]=[X,fgY]-f[X,gY]+f[X,gY]-fg[X,Y] \\ &= A(X,f,gY)+fA(X,g,Y)=gA(X,f,Y)+fA(X,g,Y)\,, \end{aligned} \tag{26}$$

thus the derivation property. This implies that A is represented by an $\mathcal{A}$-module homomorphism $b^l:\mathcal{E}\to\mathsf{Der}(\mathcal{A})\otimes_{\mathcal{A}}\mathsf{End}(\mathcal{E})$. Analogous considerations give us the right generalized anchor map b^r.

Conversely, assume the existence of both generalized anchor maps. Then, the map A defined as above reads $A(X,f,Y)=b^l(X)(\mathrm{d}f\otimes Y)$, so is $\mathcal{A}$-linear with respect to X and Y. Hence,

$$(\delta_1(f)\delta_2(g)B)(X,Y)=A(fX,g,Y)-fA(X,g,Y)=0$$

and

$$(\delta_2(f)\delta_2(g)B)(X,Y)=A(X,g,fY)-fA(X,g,Y)=0\,.$$

A similar reasoning for b^r gives $(\delta_1(f)\delta_1(g)B)(X,Y)=0$, so the bracket is a bidifferential operator of total order ≤ 1. □

In the case when we deal with a quasi algebroid, i.e., $\mathcal{A}=C^\infty(M)$ and $\mathcal{E}=\mathsf{Sec}(E)$ for a vector bundle $\tau:E\to M$, the generalized anchor maps (24) are associated with vector bundle maps that we denote (with some abuse of notations) also by b^r,b^l,

$$b^r,b^l:E\to\mathsf{T}M\otimes_M\mathsf{End}(E)\,,$$

covering the identity on M. Here, $\mathsf{End}(E)$ is the endomorphism bundle of E, so $\mathsf{End}(E)\simeq E^*\otimes_M E$. The induced maps of sections produce from sections of E sections of $\mathsf{T}M\otimes_M\mathsf{End}(E)$ which, in turn, act on sections of $\mathsf{T}^*M\otimes_M E$ in the obvious way. An algebroid version of Theorem 3.4 is the following.

Theorem 3.5. *An $\mathbb{R}$-bilinear bracket $B=[\cdot,\cdot]$ on the real space $\mathsf{Sec}(E)$ of sections of a vector bundle $\tau:E\to M$ defines a quasi algebroid structure if and only if there are vector bundle morphisms*

$$b^r,b^l:E\to\mathsf{T}M\otimes_M\mathsf{End}(E) \tag{27}$$

covering the identity on M, called generalized anchor maps, right and left*, such that, for all $X,Y\in\mathsf{Sec}(E)$ and all $f\in C^\infty(M)$, (25) is satisfied. The generalized anchor maps are actual anchor maps, if they take values in $\mathsf{T}M\otimes\langle\mathrm{Id}_E\rangle\simeq\mathsf{T}M$.*

4 Loday algebroids

Let us isolate and specify the most important particular cases of Definition 3.1.

Definition 4.1.

1. A *Kirillov-Loday algebroid* (resp., *Kirillov-Lie algebroid*) on a vector bundle E over a base manifold M is a Loday bracket (resp., a Lie bracket) on the $C^\infty(M)$-module $\mathsf{Sec}(E)$ of smooth sections of E which is a bidifferential operator.

2. A *weak Loday algebroid* (resp., *weak Lie algebroid*) on a vector bundle E over a base manifold M is a Loday bracket (resp., a Lie bracket) on the $C^\infty(M)$-module $\mathsf{Sec}(E)$ of smooth sections of E which is a bidifferential operator of degree ≤ 1 with respect to each variable separately.

3. A *Loday quasi algebroid* (resp., *Lie quasi algebroid*) on a vector bundle E over a base manifold M is a Loday bracket (resp., Lie bracket) on the $C^\infty(M)$-module $\mathsf{Sec}(E)$ of smooth sections of E which is a bidifferential operator of total degree ≤ 1.

4. A *QD-algebroid* (resp., *skew QD-algebroid, Loday QD-algebroid, Lie QD-algebroid*) on a vector bundle E over a base manifold M is an $\mathbb{R}$-bilinear bracket (resp., skew bracket, Loday bracket, Lie bracket) on the $C^\infty(M)$-module $\mathsf{Sec}(E)$ of smooth sections of E for which the adjoint operators ad_X and ad^r_X are derivative endomorphisms.

Remark 4.2. *Lie pseudoalgebras* appeared first in the paper of Herz [18], but one can find similar concepts under more than a dozen of names in the literature (e.g. *Lie modules,* (R, A)*-Lie algebras, Lie-Cartan pairs, Lie-Rinehart algebras, differential algebras*, etc.). Lie algebroids were introduced by Pradines [41] as infinitesimal parts of differentiable groupoids. In the same year a book by Nelson was published where a general theory of Lie modules, together with a big part of the corresponding differential calculus, can be found. We also refer to a survey article by Mackenzie [33]. QD-algebroids, as well as Loday QD-algebroids and Lie QD-algebroids, have been introduced in [11]. In [17, 9] Loday strong QD-algebroids have been called Loday algebroids and strong QD-algebroids have been called just *algebroids*. The latter served as geometric framework for generalized Lagrange and Hamilton formalisms.

In the case of line bundles, $\mathrm{rk}\, E = 1$, Lie QD-algebroids are exactly *local Lie algebras* in the sense of Kirillov [24]. They are just *Jacobi brackets*, if the bundle is trivial, $\mathsf{Sec}(E) = C^\infty(M)$. Of course, Lie QD-algebroid brackets are first-order bidifferential operators by definition, while Kirillov has originally started with considering Lie brackets on sections of line bundles determined by local operators and has only later discovered that these operators have to be bidifferential operators of first order. A purely algebraic version of Kirillov's result has been proven in [10], Theorems 4.2 and 4.4, where bidifferential Lie brackets on associative commutative algebras containing no nilpotents have been considered.

Example 4.3. Let us consider a Loday algebroid bracket in the sense of [19, 20, 21, 27, 36, 48], i.e., a Loday algebra bracket $[\cdot,\cdot]$ on the $C^\infty(M)$-module $\mathcal{E} = \mathsf{Sec}(E)$ of sections of a vector bundle $\tau : E \to M$ for which there is a vector bundle morphism $\rho : E \to \mathsf{T}M$ covering the identity on M (the left anchor map) such that (1) is satisfied. Since, due to (19), the anchor map is necessarily a homomorphism of the Loday bracket into the Lie bracket of vector fields, our Loday algebroid is just a Lie algebroid in the case when ρ is injective. In the other cases the anchor map does not determine the Loday algebroid structure, in particular does not imply any locality of the bracket with respect to the first argument. Thus, this concept of Loday algebroid is not geometric.

For instance, let us consider a Whitney sum bundle $E = E_1 \oplus_M E_2$ with the canonical projections $p_i : E \to E_i$ and any $\mathbb{R}$-linear map $\varphi : \mathsf{Sec}(E_1) \to C^\infty(M)$. Being only $\mathbb{R}$-linear, φ can be chosen very strange non-geometric and non-local. Define now the following bracket on $\mathsf{Sec}(E)$:

$$[X, Y] = \varphi(p_1(X)) \cdot p_2(Y)\,.$$

It is easy to see that this is a Loday bracket which admits the trivial left anchor, but the bracket is non-local and non-geometric as well.

Example 4.4. A standard example of a weak Lie algebroid bracket is a Poisson (or, more generally, Jacobi) bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ viewed as a $C^\infty(M)$-module of section of the trivial line bundle $M\times\mathbb{R}$. It is a bidifferential operator of order ≤ 1 and the total order ≤ 2. It is actually a Lie QD-algebroid bracket, as ad_f and ad^r_f are, by definition, derivations (more generally, first-order differential operators). Both anchor maps coincide and give the corresponding Hamiltonian vector fields, $\rho(f)(g) = \{f, g\}$. The map $f \mapsto \rho(f)$ is again a differential operator of order 1, so is not implemented by a vector bundle morphism $\rho : M \times \mathbb{R} \to \mathsf{T}M$. Therefore, this weak Lie algebroid is not a Lie algebroid.

Example 4.5. Various brackets are associated with a volume form ω on a manifold M of dimension n (see e.g. [28]). Denote with $\mathcal{X}^k(M)$ (resp., $\Omega^k(M)$) the spaces of k-vector fields (resp., k-forms) on M. As the contraction maps $\mathcal{X}^k(M) \ni K \mapsto i_K\omega \in \Omega^{n-k}(M)$ are isomorphisms of $C^\infty(M)$-modules, to the de Rham cohomology operator $\mathrm{d} : \Omega^{n-k-1}(M) \to \Omega^{n-k}(M)$ corresponds a homology operator $\delta : \mathcal{X}^k(M) \to \mathcal{X}^{k-1}(M)$. The skew-symmetric bracket B on $\mathcal{X}^2(M)$ defined in [28] by $B(t, u) = -\delta(t) \wedge \delta(u)$ is not a Lie bracket, since its Jacobiator $B(B(t, u), v) + c.p.$ equals $\delta(\delta(t)\wedge\delta(u)\wedge\delta(v))$. A solution proposed in [28] depends on considering the algebra N of bivector fields modulo δ-exact bivector fields for which the Jacobi anomaly disappears, so that N is a Lie algebra.

Another option is to resign from skew-symmetry and define the corresponding Kirillov-Loday algebroid. In view of the duality between $\mathcal{X}^2(M)$ and Ω^{n-2}, it is possible to work with $\Omega^{n-2}(M)$ instead. For $\gamma \in \Omega^{n-2}(M)$ we define the vector field $\widehat{\gamma} \in \mathcal{X}(M)$ from the formula $i_{\widehat{\gamma}}\omega = \mathrm{d}\gamma$. The bracket in $\Omega^{n-2}(M)$ is now defined by

$$\{\gamma, \beta\}_\omega = \pounds_{\widehat{\gamma}}\beta = i_{\widehat{\gamma}} i_{\widehat{\beta}}\omega + \mathrm{d} i_{\widehat{\gamma}}\beta\,.$$

Since we have

$$i_{[\widehat{\gamma},\widehat{\beta}]_{vf}}\omega = \pounds_{\widehat{\gamma}} i_{\widehat{\beta}}\omega - i_{\widehat{\beta}}\pounds_{\widehat{\gamma}}\omega = \mathrm{d} i_{\widehat{\gamma}} i_{\widehat{\beta}}\omega = \mathrm{d}\{\gamma, \beta\}_\omega\,,$$

it holds

$$\widehat{\{\gamma, \beta\}_\omega} = [\widehat{\gamma}, \widehat{\beta}]_{vf}\,.$$

Therefore,

$$\{\{\gamma, \beta\}_\omega, \eta\}_\omega = \pounds_{\widehat{\{\gamma,\beta\}_\omega}}\eta = \pounds_{\widehat{\gamma}}\pounds_{\widehat{\beta}}\eta - \pounds_{\widehat{\beta}}\pounds_{\widehat{\gamma}}\eta = \{\gamma, \{\beta, \eta\}_\omega\}_\omega - \{\beta, \{\gamma, \eta\}_\omega\}_\omega\,,$$

so the Jacobi identity is satisfied and we deal with a Loday algebra. This is in fact a Kirillov-Loday algebroid structure on $\wedge^{n-2}\mathsf{T}^*M$ with the left anchor $\rho(\gamma) = \widehat{\gamma}$. This bracket is a bidifferential operator which is first-order with respect to the second argument and second-order with respect to the first one.

Note that Lie QD-algebroids are automatically Lie algebroids, if the rank of the bundle E is > 1 [11, Theorem 3]. Also some other of the above concepts do not produce qualitatively new examples.

Theorem 4.6. *([11, 13, 14])*

(a) Any Loday bracket on $C^\infty(M)$ (more generally, on sections of a line bundle) which is a bidifferential operator is actually a Jacobi bracket (first-order and skew-symmetric).

(b) Let $[\cdot,\cdot]$ be a Loday bracket on sections of a vector bundle $\tau : E \to M$, admitting anchor maps $\rho, \rho^r : \mathsf{Sec}(E) \to \mathcal{X}(M)$ which assign vector fields to sections of E and such that (21) is satisfied (Loday QD-algebroid on E). Then, the anchors coincide, $\rho = \rho^r$, and the bracket is skew-symmetric at points $p \in M$ in the support of $\rho = \rho^r$. Moreover, if the rank of E is > 1, then the anchor maps are $C^\infty(M)$-linear, i.e. they come from a vector bundle morphism $\rho = \rho^r : E \to M$. In other words, any Loday QD-algebroid is actually, around points where one anchor does not vanish, a Jacobi bracket if $\mathrm{rk}(E) = 1$*, or Lie algebroid bracket if* $\mathrm{rk}(E) > 1$.

The above results show that relaxing skew-symmetry and considering Loday brackets on $C^\infty(M)$ or $\mathsf{Sec}(E)$ does not lead to new structures (except for just bundles of Loday algebras), if we assume differentiability in the first case and the existence of both (possibly different) anchor maps in the second. Therefore, a definition of Loday algebroids that admits a rich family of new examples, must resign from the traditionally understood right anchor map.

The definition of the main object of our studies can be formulated as follows.

Definition 4.7. A *Loday algebroid* on a vector bundle E over a base manifold M is a Loday bracket on the $C^\infty(M)$-module $\mathsf{Sec}(E)$ of smooth sections of E which is a bidifferential operator of total degree ≤ 1 and for which the adjoint operator ad_X is a derivative endomorphism.

Theorem 4.8. *A Loday bracket $[\cdot,\cdot]$ on the real space $\mathsf{Sec}(E)$ of sections of a vector bundle $\tau : E \to M$ defines a Loday algebroid structure if and only if there are vector bundle morphisms*

$$\rho : E \to \mathsf{T}M\,, \quad \alpha : E \to \mathsf{T}M \otimes_M \mathsf{End}(E)\,, \tag{28}$$

covering the identity on M, such that, for all $X, Y \in \mathsf{Sec}(E)$ and all $f \in C^\infty(M)$,

$$[X, fY] = f[X,Y] + \rho(X)(f)Y\,, \quad [fX, Y] = f[X,Y] - \rho(Y)(f)X + \alpha(Y)(\mathrm{d}f \otimes X)\,. \tag{29}$$

If this is the case, the left anchor induces a homomorphism of the Loday bracket into the bracket $[\cdot,\cdot]_{vf}$ of vector fields,

$$\rho([X,Y]) = [\rho(X), \rho(Y)]_{vf}\,.$$

Proof. This is a direct consequence of Theorem 3.5 and the fact that an algebroid bracket has the left anchor map. We just write the generalized right anchor map as $b^r = \rho \otimes I - \alpha$. □

To give a local form of a Loday algebroid bracket, let us recall that sections X of the vector bundle E can be identified with linear (along fibers) functions ι_X on the dual bundle E^*. Thus, fixing local coordinates (x^a) in M and a basis of local sections e_i of E, we have a corresponding system $(x^a, \xi_i = \iota_{e_i})$ of affine coordinates in E^*. As local sections of E are identified with linear functions $\sigma = \sigma^i(x)\xi_i$, the Loday bracket is represented by a bidifferential operator B of total order ≤ 1:

$$B(\sigma_1^i(x)\xi_i, \sigma_2^j(x)\xi_j) = c_{ij}^k(x)\sigma_1^i(x)\sigma_2^j(x)\xi_k + \beta_{ij}^{ak}(x)\frac{\partial \sigma_1^i}{\partial x^a}(x)\sigma_2^j(x)\xi_k + \gamma_{ij}^{ak}(x)\sigma_1^i(x)\frac{\partial \sigma_2^j}{\partial x^a}(x)\xi_k\,.$$

Taking into account the existence of the left anchor, we have

$$\begin{aligned} B(\sigma_1^i(x)\xi_i, \sigma_2^j(x)\xi_j) \;&=\; c_{ij}^k(x)\sigma_1^i(x)\sigma_2^j(x)\xi_k + \alpha_{ij}^{ak}(x)\frac{\partial \sigma_1^i}{\partial x^a}(x)\sigma_2^j(x)\xi_k \\ &\quad + \rho_i^a(x)\left(\sigma_1^i(x)\frac{\partial \sigma_2^j}{\partial x^a}(x) - \frac{\partial \sigma_1^j}{\partial x^a}(x)\sigma_2^i(x)\right)\xi_j\,. \end{aligned} \tag{30}$$

Since sections of $\mathsf{End}(E)$ can be written in the form of linear differential operators, we can rewrite (30) in the form

$$B = c_{ij}^k(x)\xi_k\partial_{\xi_i}\otimes\partial_{\xi_j} + \alpha_{ij}^{ak}(x)\xi_k\partial_{x^a}\partial_{\xi_i}\otimes\partial_{\xi_j} + \rho_i^a(x)\partial_{\xi_i}\wedge\partial_{x^a}\,. \tag{31}$$

Of course, there are additional relations between coefficients of B due to the fact that the Jacobi identity is satisfied.

5 Examples

5.1 Leibniz algebra

Of course, a finite-dimensional Leibniz algebra is a Leibniz algebroid over a point.

5.2 Courant-Dorfman bracket

The *Courant bracket* is defined on sections of $\mathcal{T}M = \mathsf{T}M\oplus_M \mathsf{T}^*M$ as follows:

$$[X+\omega, Y+\eta] = [X,Y]_{vf} + \pounds_X\eta - \pounds_Y\omega - \frac{1}{2}\left(d\,i_X\eta - d\,i_Y\omega\right)\,. \tag{32}$$

This bracket is antisymmetric, but it does not satisfy the Jacobi identity; the Jacobiator is an exact 1-form. It is, as easily seen, given by a bidifferential operator of total order ≤ 1, so it is a skew quasi algebroid.

The *Dorfman bracket* is defined on the same module of sections. Its definition is the same as for Courant, except that the corrections and the exact part of the second Lie derivative disappear:

$$[X+\omega, Y+\eta] = [X,Y]_{vf} + \pounds_X\eta - i_Y\,\mathrm{d}\omega = [X,Y]_{vf} + i_X\,\mathrm{d}\eta - i_Y\,\mathrm{d}\omega + \mathrm{d}\,i_X\eta\,. \tag{33}$$

This bracket is visibly non skew-symmetric, but it is a Loday bracket which is bidifferential of total order ≤ 1. Moreover, the Dorfman bracket admits the classical left anchor map

$$\rho : \mathcal{T}M = \mathsf{T}M\oplus_M\mathsf{T}^*M \to \mathsf{T}M \tag{34}$$

which is the projection onto the first component. Indeed,

$$[X+\omega, f(Y+\eta)] = [X,fY]_{vf} + \pounds_X f\eta - i_{fY}\,\mathrm{d}\omega = f[X+\omega,Y+\eta] + X(f)(Y+\eta)\,.$$

For the right generalized anchor we have

$$\begin{aligned}[f(X+\omega), Y+\eta] &= [fX,Y]_{vf} + i_{fX}\,\mathrm{d}\eta - i_Y\,\mathrm{d}(f\omega) + \mathrm{d}\,i_{fX}\eta \\ &= f[X+\omega,Y+\eta] - Y(f)(X+\omega) + \mathrm{d}f\wedge(i_X\eta + i_Y\omega)\,,\end{aligned}$$

so that

$$\alpha(Y+\eta)(\mathrm{d}f\otimes(X+\omega)) = \mathrm{d}f\wedge(i_X\eta+i_Y\omega) = 2\langle X+\omega, Y+\eta\rangle_+\cdot\mathrm{d}f\,,$$

where

$$\langle X+\omega, Y+\eta\rangle_+ = \frac{1}{2}\left(i_X\eta + i_Y\omega\right) = \frac{1}{2}\left(\langle X,\eta\rangle + \langle Y,\omega\rangle\right)\,,$$

is a symmetric nondegenerate bilinear form on $\mathcal{T}M$ (while $\langle\cdot,\cdot\rangle$ is the canonical pairing). We will refer to it, though it is not positively defined, as the *scalar product* in the bundle $\mathcal{T}M$.

Note that $\alpha(Y+\eta)$ is really a section of $\mathsf{T}M\otimes_M\mathsf{End}(\mathsf{T}M\oplus_M\mathsf{T}^*M)$ that in local coordinates reads

$$\alpha(Y+\eta) = \sum_k \partial_{x^k}\otimes\left(\mathrm{d}x^k\wedge(i_\eta + i_Y)\right)\,.$$

Hence, the Dorfman bracket is a Loday algebroid bracket.

It is easily checked that the Courant bracket is the antisymmetrization of the Dorfman bracket, and that the Dorfman bracket is the Courant bracket plus $\mathrm{d}\langle X+\omega, Y+\eta\rangle_+$

5.3 Twisted Courant-Dorfman bracket

The Courant-Dorfman bracket can be twisted by adding a term associated with a 3-form Θ [44]:

$$[X+\omega, Y+\eta] = [X,Y]_{vf} + \pounds_X\eta - i_Y\,\mathrm{d}\omega + i_{X\wedge Y}\Theta\,. \tag{35}$$

It turns out that this bracket is still a Loday bracket if the 3-form Θ is closed. As the added term is $C^\infty(M)$-linear with respect to X and Y, the anchors remain the same, thus we deal with a Loday algebroid.

5.4 Courant algebroid

Courant algebroids – structures generalizing the Courant-Dorfman bracket on $\mathcal{T}M$ – were introduced as as double objects for Lie bialgebroids by Liu, Weinstein and Xu [32] in a bit complicated way. It was shown by Roytenberg [42] that a Courant algebroid can be equivalently defined as a vector bundle $\tau : E \to M$ with a Loday bracket on $\mathsf{Sec}(E)$, an anchor $\rho : E \to \mathsf{T}M$, and a symmetric nondegenerate inner product $(\cdot,\cdot)$ on E, related by a set of four additional properties. It was further observed [46, 15] that the number of independent conditions can be reduced.

Definition 5.1. A *Courant algebroid* is a vector bundle $\tau : E \to M$ equipped with a Leibniz bracket $[\cdot,\cdot]$ on $\mathsf{Sec}(E)$, a vector bundle map (over the identity) $\rho : E \to \mathsf{T}M$, and a nondegenerate symmetric bilinear form (scalar product) $(\cdot|\cdot)$ on E satisfying the identities

$$\rho(X)(Y|Y) = 2(X|[Y,Y]), \tag{36}$$
$$\rho(X)(Y|Y) = 2([X,Y]|Y). \tag{37}$$

Note that (36) is equivalent to

$$\rho(X)(Y|Z) = (X|[Y,Z]+[Z,Y]). \tag{38}$$

Similarly, (37) easily implies the invariance of the pairing $(\cdot,\cdot)$ with respect to the adjoint maps

$$\rho(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z]), \tag{39}$$

which in turn shows that ρ is the anchor map for the left multiplication:

$$[X, fY] = f[X,Y] + \rho(X)(f)Y\,. \tag{40}$$

Twisted Courant-Dorfman brackets are examples of Courant algebroid brackets with $(\cdot,\cdot) = \langle\cdot,\cdot\rangle_+$ as the scalar product. Defining a derivation $\mathsf{D} : C^\infty(M) \to \mathsf{Sec}(E)$ by means of the scalar product

$$(\mathsf{D}(f)|X) = \frac{1}{2}\rho(X)(f)\,, \tag{41}$$

we get out of (38) that

$$[Y,Z]+[Z,Y] = 2\mathsf{D}(Y|Z)\,. \tag{42}$$

This, combined with (40), implies in turn

$$\alpha(Z)(\mathrm{d}f\otimes Y) = 2(Y|Z)\mathsf{D}(f)\,, \tag{43}$$

so any Courant algebroid is a Loday algebroid.

5.5 Grassmann-Dorfman bracket

The Dorfman bracket (33) can be immediately generalized to a bracket on sections of $\mathcal{T}^\wedge M = \mathsf{T}M \oplus_M \wedge\mathsf{T}^*M$, where

$$\wedge\mathsf{T}^*M = \bigoplus_{k=0}^{\infty} \wedge^k\mathsf{T}^*M\,,$$

so that the module of sections, $\mathsf{Sec}(\wedge\mathsf{T}^*M) = \Omega(M) = \bigoplus_{k=0}^{\infty}\Omega^k(M)$, is the Grassmann algebra of differential forms. The bracket, *Grassmann-Dorfman bracket*, is formally given by the same formula (33) and the proof that it is a Loday algebroid bracket is almost the same. The left anchor is the projection on the summand $\mathsf{T}M$,

$$\rho : \mathsf{T}M \oplus_M \wedge\mathsf{T}^*M \to \mathsf{T}M\,, \tag{44}$$

and

$$\alpha(Y+\eta)(\mathrm{d}f \otimes (X+\omega)) = \mathrm{d}f \wedge (i_X\eta + i_Y\omega) = 2\,\mathrm{d}f \wedge \langle X+\omega, Y+\eta\rangle_+\,,$$

where

$$\langle X+\omega, Y+\eta\rangle_+ = \frac{1}{2}\left(i_X\eta + i_Y\omega\right)\,,$$

is a symmetric nondegenerate bilinear form on $\mathcal{T}^\wedge M$, this time with values in $\Omega(M)$. Like for the classical Courant-Dorfman bracket, the graph of a differential form β is an isotropic subbundle in $\mathcal{T}^\wedge M$ which is involutive (its sections are closed with respect to the bracket) if and only if $\mathrm{d}\beta = 0$. The Grassmann-Dorfman bracket induces Loday algebroid brackets on all bundles $\mathsf{T}M \oplus_M \wedge^k\mathsf{T}^*M$, $k = 0,1,\dots,\infty$. These brackets have been considered in [2] and called there *higher-order Courant brackets*. Note that this is exactly the bracket derived from the bracket of first-order (super)differential operators on the Grassmann algebra $\Omega(M)$: we associate with $X+\omega$ the operator $S_{X+\omega} = i_X + \omega\wedge$ and compute the super-commutators,

$$[[S_{X+\omega}, \mathrm{d}]_{sc}, S_{Y+\eta}]_{sc} = S_{[X+\omega, Y+\eta]}\,.$$

5.6 Grassmann-Dorfman bracket for a Lie algebroid

All the above remains valid when we replace $\mathsf{T}M$ with a Lie algebroid $(E, [\cdot,\cdot]_E, \rho_E)$, the de Rham differential d with the Lie algebroid cohomology operator d^E on $\mathsf{Sec}(\wedge E^*)$, and the Lie derivative along vector fields with the Lie algebroid Lie derivative $\pounds^E$. We define a bracket on sections of $E \oplus_M \wedge E^*$ with formally the same formula

$$[X+\omega, Y+\eta] = [X,Y]_E + \pounds^E_X\eta - i_Y\,\mathrm{d}^E\omega\,. \tag{45}$$

This is a Loday algebroid bracket with the left anchor

$$\rho : E \oplus_M \wedge E^* \to \mathsf{T}M\,, \quad \rho(X+\omega) = \rho_E(X)$$

and

$$\alpha(Y+\eta)(\mathrm{d}f \otimes (X+\omega)) = \mathrm{d}^E f \wedge (i_X\eta + i_Y\omega)\,.$$

5.7 Lie derivative bracket for a Lie algebroid

The above Loday bracket on sections of $E \oplus_M \wedge E^*$ has a simpler version. Let us put simply

$$[X+\omega, Y+\eta] = [X,Y]_E + \pounds^E_X\eta\,. \tag{46}$$

This is again a Loday algebroid bracket with the same left anchor and and

$$\alpha(Y+\eta)(\mathrm{d}f \otimes (X+\omega)) = \mathrm{d}^E f \wedge i_X\eta + \rho_E(Y)(f)\omega\,.$$

In particular, when reducing to 0-forms, we get a Leibniz algebroid structure on $E\times\mathbb{R}$, where the bracket is defined by $[X+f,Y+g]=[X,Y]_E+\rho_E(X)g$, the left anchor by $\rho(X,f)=\rho_E(X)$, and the generalized right anchor by

$$b^r(Y,g)(\mathrm{d}h\otimes(X+f))=-\rho_E(Y)(h)X\,.$$

In other words,

$$\alpha(Y,g)(\mathrm{d}h\otimes(X+f))=\rho_E(Y)(h)f\,.$$

5.8 Loday algebroids associated with a Nambu-Poisson structure

In the following M denotes a smooth m-dimensional manifold and n is an integer such that $3\leq n\leq m$. An almost Nambu-Poisson structure of order n on M is an n-linear bracket $\{\cdot,\dots,\cdot\}$ on $C^\infty(M)$ that is skew-symmetric and has the Leibniz property with respect to the point-wise multiplication. It corresponds to an n-vector field $\Lambda\in\Gamma(\wedge^n\mathsf{T}M)$. Such a structure is Nambu-Poisson if it verifies the *Filippov identity* (*generalized Jacobi identity*):

$$\begin{gathered}\{f_1,\dots,f_{n-1},\{g_1,\dots,g_n\}\}=\{\{f_1,\dots,f_{n-1},g_1\},g_2,\dots,g_n\}+\\ \{g_1,\{f_1,\dots,f_{n-1},g_2\},g_3,\dots,g_n\}+\cdots+\{g_1,\dots,g_{n-1},\{f_1,\dots,f_{n-1},g_n\}\}\,,\end{gathered}\tag{47}$$

i.e., if the Hamiltonian vector fields $X_{f_1\dots f_{n-1}}=\{f_1,\dots,f_{n-1},\cdot\}$ are derivations of the bracket. Alternatively, an almost Nambu-Poisson structure is Nambu-Poisson if and only if

$$\pounds_{X_{f_1,\dots,f_{n-1}}}\Lambda=0\,,$$

for all functions $f_1,\dots,f_{n-1}$.

Spaces equipped with skew-symmetric brackets satisfying the above identity have been introduced by Filippov [6] under the name *n-Lie algebras*.

The concept of Leibniz (Loday) algebroid used in [21] is the usual one, without differentiability condition for the first argument. Actually, this example is a Loday algebroid in our sense as well. The bracket is defined for $(n-1)$-forms by

$$[\omega,\eta]=\pounds_{\rho(\omega)}\eta+(-1)^n(i_{\mathrm{d}\omega}\Lambda)\eta\,,$$

where

$$\rho:\wedge^{n-1}\mathsf{T}^*M\ni\omega\mapsto i_\omega\Lambda\in\mathsf{T}M$$

is actually the left anchor. Indeed,

$$[\omega,f\eta]=\pounds_{\rho(\omega)}f\eta+(-1)^n(i_{\mathrm{d}\omega}\Lambda)f\eta=f[\omega,\eta]+\rho(\omega)(f)\eta\,.$$

For the generalized right anchor we get

$$[f\omega,\eta]=\pounds_{\rho(f\omega)}\eta+(-1)^n(i_{\mathrm{d}(f\omega)}\Lambda)\eta=f[\omega,\eta]-i_{\rho(\omega)}(\mathrm{d}f\wedge\eta)\,,$$

so

$$\alpha(\eta)(\mathrm{d}f\otimes\omega)=\rho(\eta)(f)\,\omega-\rho(\omega)(f)\,\eta+\mathrm{d}f\wedge i_{\rho(\omega)}\eta\,.$$

Note that α is really a bundle map $\alpha:\wedge^{n-1}\mathsf{T}^*M\to\mathsf{T}M\otimes_M\mathsf{End}(\wedge^{n-1}\mathsf{T}^*M)$, since it is obviously $C^\infty(M)$-linear in η and ω, as well as a derivation with respect to f.

In [19, 20], another Leibniz algebroid associated with the Nambu-Poisson structure Λ is proposed. The vector bundle is the same, $E=\wedge^{n-1}\mathsf{T}^*M$, the left anchor map is the same as well, $\rho(\omega)=i_\omega\Lambda$, but the Loday bracket reads

$$[\omega,\eta]'=\pounds_{\rho(\omega)}\eta-i_{\rho(\eta)}\mathrm{d}\omega\,.$$

Hence,

$$\begin{aligned}[][f\omega,\eta]' &= \pounds_{\rho(f\omega)}\eta - i_{\rho(\eta)}\mathrm{d}(f\omega)\\ &= f[\omega,\eta]' - \rho(\eta)(f)\,\omega + \mathrm{d}f\wedge(i_{\rho(\omega)}\eta + i_{\rho(\eta)}\omega)\,,\end{aligned}$$

so that for the generalized right anchor we get

$$\alpha(\eta)(\mathrm{d}f\otimes\omega) = \mathrm{d}f\wedge(i_{\rho(\omega)}\eta + i_{\rho(\eta)}\omega)\,.$$

This Loday algebroid structure is clearly the one obtained from the Grassmann-Dorfman bracket on the graph of Λ,

$$\mathrm{graph}(\Lambda) = \{\rho(\omega)+\omega : \omega\in\Omega^{n-1}(M)\}\,.$$

Actually, an n-vector field Λ is a Nambu-Poisson tensor if and only if its graph is closed with respect to the Grassmann-Dorfman bracket [2].

6 The Lie pseudoalgebra of a Loday algebroid

Let us fix a Loday pseudoalgebra bracket $[\cdot,\cdot]$ on an $\mathcal{A}$-module $\mathcal{E}$. Let $\rho:\mathcal{E}\to\mathsf{Der}(\mathcal{A})$ be the left anchor map, and let

$$b^r = \rho-\alpha : \mathcal{E}\to\mathsf{Der}(\mathcal{A})\otimes_{\mathcal{A}}\mathsf{End}(\mathcal{E})$$

be the generalized right anchor map. For every $X\in\mathcal{E}$ we will view $\alpha(X)$ as a $\mathcal{A}$-module homomorphism $\alpha(X):\Omega^1\otimes_{\mathcal{A}}\mathcal{E}\to\mathcal{E}$, where Ω^1 is the $\mathcal{A}$-submodule of $\mathsf{Hom}_{\mathcal{A}}(\mathcal{E};\mathcal{A})$ generated by $\mathrm{d}\mathcal{A}=\{\mathrm{d}f : f\in\mathcal{A}\}$ and $\mathrm{d}f(D)=D(f)$.

It is a well-known fact that the subspace $\mathfrak{g}^0$ generated in a Loday algebra $\mathfrak{g}$ by the symmetrized brackets $X\diamond Y=[X,Y]+[Y,X]$ is a two-sided ideal and that $\mathfrak{g}/\mathfrak{g}^0$ is a Lie algebra. Putting

$$\mathcal{E}^0 = \mathrm{span}\{[X,X] : X\in\mathcal{E}\}\,,$$

we have then

$$[\mathcal{E}^0,\mathcal{E}]=0\,,\quad [\mathcal{E},\mathcal{E}^0]\subset\mathcal{E}^0\,. \tag{48}$$

Indeed, symmetrized brackets are spanned by squares $[X,X]$, so, due to the Jacobi identity,

$$[[X,X],Y]=[X,[X,Y]]-[X,[X,Y]]=0$$

and

$$[Y,[X,X]]=[[Y,X],X]+[X,[Y,X]]=[X,Y]\diamond Y\,. \tag{49}$$

However, working with $\mathcal{A}$-modules, we would like to have an $\mathcal{A}$-module structure on $\mathcal{E}/\mathcal{E}^0$. Unfortunately, $\mathcal{E}^0$ is not a submodule in general. Let us consider therefore the $\mathcal{A}$-submodule $\bar{\mathcal{E}}^0$ of $\mathcal{E}$ generated by $\mathcal{E}^0$, i.e., $\bar{\mathcal{E}}^0=\mathcal{A}\cdot\mathcal{E}^0$.

Lemma 6.1. *For all $f\in\mathcal{A}$ and $X,Y,Z\in\mathcal{E}$ we have*

$$\alpha(X)(\mathrm{d}f\otimes Y) = X\diamond(fY)-f(X\diamond Y)\,, \tag{50}$$

$$[\alpha(X)(\mathrm{d}f\otimes Y),Z] = \rho(Z)(f)(X\diamond Y)-\alpha(Z)(\mathrm{d}f\otimes(X\diamond Y))\,. \tag{51}$$

In particular,

$$[\alpha(X)(\mathrm{d}f\otimes Y),Z]=[\alpha(Y)(\mathrm{d}f\otimes X),Z]\,. \tag{52}$$

Proof. To prove (50) it suffices to combine the identity $[X,fY]=f[X,Y]+\rho(X)(f)(Y)$ with

$$[fY,X]=f[Y,X]-\rho(X)(f)Y+\alpha(X)(\mathrm{d}f\otimes Y)\,.$$

Then, as $[\mathcal{E}^0,\mathcal{E}]=0$,

$$[\alpha(X)(\mathrm{d}f\otimes Y),Z]=-[f(X\diamond Y),Z]=\rho(Z)(f)(X\diamond Y)-\alpha(Z)(\mathrm{d}f\otimes(X\diamond Y))\,.$$

□

Corollary 6.2. *For all $f\in\mathcal{A}$ and $X,Y\in\mathcal{E}$,*

$$\alpha(X)(\mathrm{d}f\otimes Y)\in\bar{\mathcal{E}}^0\,, \tag{53}$$

and the left anchor vanishes on $\bar{\mathcal{E}}^0$,

$$\rho(\bar{\mathcal{E}}^0)=0\,. \tag{54}$$

Moreover, $\bar{\mathcal{E}}^0$ is a two-sided Loday ideal in $\mathcal{E}$ and the Loday bracket induces on the $\mathcal{A}$-module $\bar{\mathcal{E}}=\mathcal{E}/\bar{\mathcal{E}}^0$ a Lie pseudoalgebra structure with the anchor

$$\bar{\rho}([X])=\rho(X)\,, \tag{55}$$

where $[X]$ denotes the coset of X.

Proof. The first statement follows directly from (50). As $[\mathcal{E}^0,\mathcal{E}]=0$, the anchor vanishes on $\mathcal{E}^0$ and thus on $\bar{\mathcal{E}}^0=\mathcal{A}\cdot\mathcal{E}^0$. From

$$[Z,f(X\diamond Y)]=f[Z,X\diamond Y]+\rho(X)(f)(X\diamond Y)\in\bar{\mathcal{E}}^0$$

and

$$[f(X\diamond Y),Z]=f[(X\diamond Y),Z]-\rho(Z)(f)(X\diamond Y)+\alpha(Z)(\mathrm{d}f\otimes(X\diamond Y))\in\bar{\mathcal{E}}^0\,,$$

we conclude that $\bar{\mathcal{E}}^0$ is a two-sided ideal. As $\bar{\mathcal{E}}^0$ contains all elements $X\diamond Y$, The Loday bracket induces on $\mathcal{E}/\bar{\mathcal{E}}^0$ a skew-symmetric bracket with the anchor (55) and satisfying the Jacobi identity, thus a Lie pseudoalgebra structure. □

Definition 6.3. The Lie pseudoalgebra $\bar{\mathcal{E}}=\mathcal{E}/\bar{\mathcal{E}}^0$ we will call the *Lie pseudoalgebra of the Loday pseudoalgebra* $\mathcal{E}$. If $\mathcal{E}=\mathsf{Sec}(E)$ is the Loday pseudoalgebra of a Loday algebroid on a vector bundle E and the module $\bar{\mathcal{E}}^0$ is the module of sections of a vector subbundle $\bar{E}$ of E, we deal with the *Lie algebroid of the Loday algebroid E*.

Example 6.4. The Lie algebroid of the Courant-Dorfman bracket is the canonical Lie algebroid $\mathsf{T}M$.

Theorem 6.5. *For any Loday pseudoalgebra structure on an $\mathcal{A}$-module $\mathcal{E}$ there is a short exact sequence of morphisms of Loday pseudoalgebras over $\mathcal{A}$,*

$$0\longrightarrow\bar{\mathcal{E}}^0\longrightarrow\mathcal{E}\longrightarrow\bar{\mathcal{E}}\longrightarrow 0\,, \tag{56}$$

where $\bar{\mathcal{E}}^0$ – the $\mathcal{A}$-submodule in $\mathcal{E}$ generated by $\{[X,X]:X\in\mathcal{E}\}$ – is a Loday pseudoalgebra with the trivial left anchor and $\bar{\mathcal{E}}=\mathcal{E}/\bar{\mathcal{E}}^0$ is a Lie pseudoalgebra.

Note that the Loday ideal $\mathcal{E}^0$ is clearly commutative, while the modular ideal $\bar{\mathcal{E}}^0$ is no longer commutative in general.

7 Loday algebroid cohomology

We first recall the definition of the Loday cochain complex associated to a bi-module over a Loday algebra [31].

Let $\mathbb{K}$ be a field of nonzero characteristic and V a $\mathbb{K}$-vector space endowed with a (left) Loday bracket $[\cdot,\cdot]$. A *bimodule* over a Loday algebra $(V,[\cdot,\cdot])$ is a $\mathbb{K}$-vector space W together with a left (resp., right) *action* $\mu^l \in \mathsf{Hom}(V\otimes W, W)$ (resp., $\mu^r \in \mathsf{Hom}(W\otimes V, W)$) that verify the following requirements

$$\mu^r[x,y] = \mu^r(y)\mu^r(x) + \mu^l(x)\mu^r(y), \tag{57}$$

$$\mu^r[x,y] = \mu^l(x)\mu^r(y) - \mu^r(y)\mu^l(x), \tag{58}$$

$$\mu^l[x,y] = \mu^l(x)\mu^l(y) - \mu^l(y)\mu^l(x), \tag{59}$$

for all $x,y\in V$.

The *Loday cochain complex* associated to the Loday algebra $(V,[\cdot,\cdot])$ and the bimodule (W,μ^l,μ^r), shortly – to $B=([\cdot,\cdot],\mu^r,\mu^l)$, is made up by the cochain space

$$\mathsf{Lin}^\bullet(V,W) = \bigoplus_{p\in\mathbb{N}} \mathsf{Lin}^p(V,W) = \bigoplus_{p\in\mathbb{N}} \mathsf{Hom}(V^{\otimes p}, W),$$

where we set $\mathsf{Lin}^0(V,W)=W$, and the coboundary operator ∂_B defined, for any p-cochain c and any vectors $x_1,\ldots,x_{p+1}\in V$, by

$$\begin{aligned}(\partial_B c)(x_1,\ldots,x_{p+1}) \quad &= \quad (-1)^{p+1}\mu^r(x_{p+1})c(x_1,\ldots,x_p) + \sum_{i=1}^{p}(-1)^{i+1}\mu^l(x_i)c(x_1,\ldots\hat{\imath}\ldots,x_{p+1}) \\ &\quad + \sum_{i<j}(-1)^i c(x_1,\ldots\hat{\imath}\ldots,\overbrace{[x_i,x_j]}^{(j)},\ldots,x_{p+1}) \ .\end{aligned} \tag{60}$$

Let now ρ be a *representation* of the Loday algebra $(V,[\cdot,\cdot])$ on a $\mathbb{K}$-vector space W, i.e. a Loday algebra homomorphism $\rho: V\to \mathsf{End}(W)$. It is easily checked that $\mu^l := \rho$ and $\mu^r := -\rho$ endow W with a bimodule structure over V. Moreover, in this case of a bimodule induced by a representation, the Loday cohomology operator reads

$$\begin{aligned}(\partial_B c)(x_1,\ldots,x_{p+1}) \quad &= \quad \sum_{i=1}^{p+1}(-1)^{i+1}\rho(x_i)c(x_1,\ldots\hat{\imath}\ldots,x_{p+1}) \\ &\quad + \sum_{i<j}(-1)^i c(x_1,\ldots\hat{\imath}\ldots,\overbrace{[x_i,x_j]}^{(j)},\ldots,x_{p+1}) \ .\end{aligned} \tag{61}$$

Note that the above operator ∂_B is well defined if only the *map* $\rho: V\to\mathsf{End}(W)$ and the *bracket* $[\cdot,\cdot]: V\otimes V\to V$ are given. We will refer to it as to the *Loday operator* associated with $B=([\cdot,\cdot],\rho)$. The point is that $\partial_B^2=0$ if and only if $[\cdot,\cdot]$ is a Loday bracket and ρ is its representation. Indeed, the Loday algebra homomorphism property of ρ (resp., the Jacobi identity for $[\cdot,\cdot]$) is encoded in $\partial_B^2=0$ on $\mathsf{Lin}^0(V,W)=W$ (resp., $\mathsf{Lin}^1(V,W)$), at least if $W\neq\{0\}$, what we assume).

Let now E be a vector bundle over a manifold M. In $\mathsf{Lin}^\bullet(\mathsf{Sec}(E),C^\infty(M))$ we can distinguish the subspace $\mathcal{D}^\bullet(\mathsf{Sec}(E),C^\infty(M))\subset\mathsf{Lin}^\bullet(\mathsf{Sec}(E),C^\infty(M))$ of multidifferential operators.

If now $B = ([\cdot,\cdot],\rho)$ is an *anchored Kirollov algebroid* structure on E, i.e. $[\cdot,\cdot]$ is a Kirillov bracket (bidifferential operator) and $\rho : E \to \mathsf{T}M$ is a vector bundle morphism covering the identity, so inducing a module morphism $\rho : \mathsf{Sec}(E) \to \mathsf{Der}(C^\infty(M)) = \mathcal{X}(M)$, then it is clear that the space $\mathcal{D}^\bullet(\mathsf{Sec}(E), C^\infty(M))$ is stable under the Loday operator ∂_B associated with $B = ([\cdot,\cdot],\rho)$.

In particular, if $([\cdot,\cdot],\rho,\alpha)$ is a Loday algebroid structure on E, its left anchor $\rho : \mathsf{Sec}(E) \to \mathsf{Der}(C^\infty(M)) \subset \mathsf{End}(C^\infty(M))$ is a representation of the Loday algebra $(\mathsf{Sec}(E), [\cdot,\cdot])$ by derivations on $C^\infty(M)$ and $\partial_B^2 = 0$, so ∂_B is a coboundary operator.

Definition 7.1. Let $(E, [\cdot,\cdot], \rho, \alpha)$ be a Loday algebroid over a manifold M. We call *Loday algebroid cohomology*, the cohomology of the Loday cochain subcomplex $(\mathcal{D}^\bullet(\mathsf{Sec}(E), C^\infty(M)), \partial_B)$ associated with $B = ([\cdot,\cdot],\rho)$, i.e. the Loday algebra structure $[\cdot,\cdot]$ on $\mathsf{Sec}(E)$ represented by ρ on $C^\infty(M)$.

8 Supercommutative geometric interpretation

Let E be a vector bundle over a manifold M.

Definition 8.1. For any $\ell' \in \mathcal{D}^p(\mathsf{Sec}(E), C^\infty(M))$ and $\ell'' \in \mathcal{D}^q(\mathsf{Sec}(E), C^\infty(M))$, $p, q \in \mathbb{N}$, we define the *shuffle product*

$$(\ell' \pitchfork \ell'')(X_1, \ldots, X_{p+q}) := \sum_{\sigma \in \mathrm{sh}(p,q)} \mathrm{sign}\, \sigma \;\, \ell'(X_{\sigma_1}, \ldots, X_{\sigma_p})\, \ell''(X_{\sigma_{p+1}}, \ldots, X_{\sigma_{p+q}}),$$

where the X_i-s denote sections in $\mathsf{Sec}(E)$ and where $\mathrm{sh}(p,q) \subset \mathbb{S}_{p+q}$ is the subset of the symmetric group $\mathbb{S}_{p+q}$ made up by all (p,q)-shuffles.

The next proposition is well-known.

Proposition 8.2. *The space $\mathcal{D}^\bullet(\mathsf{Sec}(E), C^\infty(M))$ together with the shuffle multiplication $\pitchfork$ is a graded commutative associative unital $\mathbb{R}$-algebra.*

We refer to this algebra as the *shuffle algebra of the vector bundle $E \to M$*, or simply, of E. Moreover, if no confusion is possible, we write $\mathcal{D}^p(E)$ instead of $\mathcal{D}^p(\mathsf{Sec}(E), C^\infty(M))$.

Let $B = ([\cdot,\cdot],\rho)$ be an anchored Kirillov algebroid structure on E and let ∂_B be the associated Loday operator in $\mathcal{D}^\bullet(E)$. Note that we would have $\partial_B^2 = 0$ if we had assumed that we deal with a Loday algebroid.

Denote now by $\mathsf{D}^k(E)$ those k-linear multidifferential operators from $\mathcal{D}^k(E)$ which are of degree 0 with respect to the last variable and of total degree $\le k-1$, and set $\mathsf{D}^\bullet(E) = \bigcup_{k=0}^\infty \mathsf{D}^k(E)$. By convention, $\mathsf{D}^0(E) = \mathcal{D}^0(E) = C^\infty(M)$. Moreover, $\mathsf{D}^1(E) = \mathsf{Sec}(E^*)$. It is easy to see that $\mathsf{D}^\bullet(E)$ is stable for the shuffle multiplication. We will call the subalgebra $(\mathsf{D}^\bullet(E), \pitchfork)$, the *reduced shuffle algebra*, and refer to the corresponding graded ringed space as *supercommutative manifold*. Let us emphasize that this denomination is in the present text merely a terminological convention. The graded ringed spaces of the considered type are being investigated in a separate work.

Theorem 8.3. *The coboundary operator ∂_B is a degree 1 graded derivation of the shuffle algebra of E, i.e.*

$$\partial(\ell' \pitchfork \ell'') = (\partial\ell') \pitchfork \ell'' + (-1)^p \ell' \pitchfork (\partial\ell''), \tag{62}$$

for any $\ell' \in \mathcal{D}^p(E)$ and $\ell'' \in \mathcal{D}^q(E)$. Moreover, if $[\cdot,\cdot]$ is a pseudoalgebra bracket, i.e., if it is of total order ≤ 1 and ρ is the left anchor for $[\cdot,\cdot]$, then ∂_B leaves invariant the reduced shuffle algebra $\mathsf{D}^\bullet(E) \subset \mathcal{D}^\bullet(E)$.

The claim is easily checked on low degree examples. The general proof is as follows.

Proof. The value of the LHS of Equation (62) on sections $X_1, \ldots, X_{p+q+1} \in \mathsf{Sec}(E)$ is given by $S_1 + \ldots + S_4$, where

$$S_1 = \sum_{k=1}^{p+1} \sum_{\tau \in \mathrm{sh}(p,q)} (-1)^{k+1} \operatorname{sign}\tau \; \rho(X_k) \left(\ell'(X_{\tau_1}, \ldots, \widehat{X}_{\tau_k}, \ldots, X_{\tau_{p+1}}) \; \ell''(X_{\tau_{p+2}}, \ldots, X_{\tau_{p+q+1}}) \right)$$

and

$$S_3 = \sum_{1 \le k < m \le p+q+1} \sum_{\tau \in \mathrm{sh}(p,q)} (-1)^k \operatorname{sign}\tau \;\; \ell'(X_{\tau_1}, \ldots, [X_k, X_m], \ldots) \; \ell''(X_{\tau_-}, \ldots).$$

In the sum S_2, which is similar to S_1, the index k runs through $\{p+2, \ldots, p+q+1\}$ (X_{τ_k} is then missing in ℓ''). The sum S_3 contains those shuffle permutations of $1 \ldots \hat{k} \ldots p+q+1$ that send the argument $[X_k, X_m]$ with index $m =: \tau_r$ into ℓ', whereas S_4 is taken over the shuffle permutations that send $[X_k, X_m]$ into ℓ''.

Analogously, the value of $(\partial \ell') \pitchfork \ell''$ equals $T_1 + T_2$ with

$$T_1 = \sum_{\sigma \in \mathrm{sh}(p+1,q)} \sum_{i=1}^{p+1} \operatorname{sign}\sigma \, (-1)^{i+1} \left(\rho(X_{\sigma_i}) \, \ell'(X_{\sigma_1}, \ldots, \widehat{X}_{\sigma_i}, \ldots, X_{\sigma_{p+1}}) \right) \ell''(X_{\sigma_{p+2}}, \ldots, X_{\sigma_{p+q+1}})$$

and

$$T_2 = \sum_{\sigma \in \mathrm{sh}(p+1,q)} \sum_{1 \le i < j \le p+1} \operatorname{sign}\sigma \, (-1)^i \, \ell'(X_{\sigma_1}, \ldots, [X_{\sigma_i}, X_{\sigma_j}], \ldots) \; \ell''(X_{\sigma_{p+2}}, \ldots, X_{\sigma_{p+q+1}})$$

(whereas the value $T_3 + T_4$ of $(-1)^p \; \ell' \pitchfork (\partial \ell'')$, which is similar, is not (really) needed in this (sketch of) proof).

Let us stress that in S_3 and T_2 the bracket is in its natural position determined by the index $\tau_r = m$ or σ_j of its second argument, that, since $\mathrm{sh}(p,q) \simeq \mathbb{S}_{p+q}/(\mathbb{S}_p \times \mathbb{S}_q)$, the number of (p,q)-shuffles equals $(p+q)!/(p!\,q!)$, and that in S_1 the vector field $\rho(X_k)$ acts on a product of functions according to the Leibniz rule, so that each term splits. It is now easily checked that after this splitting the number of different terms in $\rho(X_-)$ (resp. $[X_-, X_-]$) in the LHS and the RHS of Equation (62) is equal to $2(p+q+1)!/(p!\,q!)$ (resp. $(p+q)(p+q+1)!/(2\,p!\,q!)$). To prove that both sides coincide, it therefore suffices to show that any term of the LHS can be found in the RHS.

We first check this for any split term of S_1 with vector field action on the value of ℓ' (the proof is similar if the field acts on the second function and also if we choose a split term in S_2),

$$(-1)^{k+1} \operatorname{sign}\tau \; \left(\rho(X_k) \ell'(X_{\tau_1}, \ldots, \widehat{X}_{\tau_k}, \ldots, X_{\tau_{p+1}}) \right) \; \ell''(X_{\tau_{p+2}}, \ldots, X_{\tau_{p+q+1}}),$$

where $k \in \{1, \ldots, p+1\}$ is fixed, as well as $\tau \in \mathrm{sh(p,q)}$ – which permutes $1 \ldots \hat{k} \ldots p+q+1$. This term exists also in T_1. Indeed, the shuffle τ induces a unique shuffle $\sigma \in \mathrm{sh}(p+1,q)$ and a unique $i \in \{1, \ldots, p+1\}$ such that $\sigma_i = k$. The corresponding term of T_1 then coincides with the chosen term in S_1, since, as easily seen, $\operatorname{sign}\sigma \; (-1)^{i+1} = (-1)^{k+1} \operatorname{sign}\tau$.

Consider now a term in S_3 (the proof is analogous for the terms of S_4),

$$(-1)^k \operatorname{sign}\tau \;\; \ell'(X_{\tau_1}, \ldots, [X_k, X_m], \ldots) \; \ell''(X_{\tau_-}, \ldots),$$

where $k < m$ are fixed in $\{1, \ldots, p+q+1\}$ and where $\tau \in \mathrm{sh}(p,q)$ is a fixed permutation of $1 \ldots \hat{k} \ldots p+q+1$ such that the section $[X_k, X_m]$ with index $m =: \tau_r$ is an argument of ℓ'.

The shuffle τ induces a unique shuffle $\sigma \in \mathrm{sh}(p+1,q)$. Set $k =: \sigma_i$ and $m =: \sigma_j$. Of course $1 \le i < j \le p+1$. This means that the chosen term reads

$$(-1)^k \operatorname{sign}\tau \;\; \ell'(X_{\sigma_1}, \ldots, [X_{\sigma_i}, X_{\sigma_j}], \ldots, X_{\sigma_{p+1}})\, \ell''(X_{\sigma_{p+2}}, \ldots, X_{\sigma_{p+q+1}}).$$

Finally this term is a term of T_2, as it is again clear that $(-1)^k \operatorname{sign}\tau = \operatorname{sign}\sigma\,(-1)^i$.

That $\mathsf{D}^\bullet(E)$ is invariant under ∂_B in the case of a pseudoalgebra bracket is obvious. This completes the proof. □

Note that the derivations ∂_B of the reduced shuffle algebra (in the case of pseudoalgebra brackets on $\mathsf{Sec}(E)$) are, due to formula (61), completely determined by their values on $\mathsf{D}^0(E) \oplus \mathsf{D}^1(E)$. More precisely, $B = ([\cdot,\cdot], \rho)$ can be easily reconstructed from ∂_B thanks to the formulae

$$\rho(X)(f) = \langle X, \partial_B f \rangle \tag{63}$$

and

$$\langle \mathfrak{l}, [X,Y] \rangle = \langle X, \partial_B \langle \mathfrak{l}, Y \rangle \rangle - \langle Y, \partial_B \langle \mathfrak{l}, X \rangle \rangle - \partial_B \mathfrak{l}(X,Y)\,, \tag{64}$$

where $X, Y \in \mathsf{Sec}(E)$, $\mathfrak{l} \in \mathsf{Sec}(E^*)$, and $f \in C^\infty(M)$.

Theorem 8.4. *If ∂ is a derivation of the reduced shuffle algebra $\mathsf{D}^\bullet(E)$, then on $\mathsf{D}^0(E) \oplus \mathsf{D}^1(E)$ the derivation ∂ coincides with ∂_B for a certain uniquely determined $B = ([\cdot,\cdot]_\partial, \rho_\partial)$ associated with a pseudoalgebra bracket $[\cdot,\cdot]_\partial$ on $\mathsf{Sec}(E)$.*

Proof. Let us define $\rho = \rho_\partial$ and $[\cdot,\cdot] = [\cdot,\cdot]_\partial$ out of formulae (63) and (64), i.e.,

$$\rho(X)(f) = \langle X, \partial f \rangle \tag{65}$$

and

$$\langle \mathfrak{l}, [X,Y] \rangle = \langle X, \partial \langle \mathfrak{l}, Y \rangle \rangle - \langle Y, \partial \langle \mathfrak{l}, X \rangle \rangle - \partial \mathfrak{l}(X,Y)\,. \tag{66}$$

The fact that $\rho(X)$ is a derivation of $C^\infty(M)$ is a direct consequence of the shuffle algebra derivation property of ∂. Eventually, the map ρ is visibly associated with a bundle map $\rho : E \to \mathsf{T}M$.

The bracket $[\cdot,\cdot]$ has ρ as left anchor. Indeed, since $\partial\mathfrak{l}(X,Y)$ is of order 0 with respect to Y, we get from (66)

$$[X, fY] - f[X,Y] = \langle X, \partial f \rangle Y = \rho(X)(f) Y\,.$$

Similarly, as $\partial\mathfrak{l}(X,Y)$ is of order 1 with respect to X and of order 0 with respect to Y, the operator

$$\delta_1(f)\,(\partial\mathfrak{l})\,(X,Y) = \partial\mathfrak{l}(fX,Y) - f\partial\mathfrak{l}(X,Y)$$

is $C^\infty(M)$-bilinear, so that the LHS of

$$\langle \mathfrak{l}, [fX,Y] - f[X,Y] \rangle = -\langle Y, \partial f \rangle \langle \mathfrak{l}, X \rangle - \delta_1(f)\,(\partial\mathfrak{l})\,(X,Y),$$

see (66), is $C^\infty(M)$-linear with respect to X and Y and a derivation with respect to f. The bracket $[\cdot,\cdot]$ is therefore of total order ≤ 1 with the generalized right anchor $b^r = \rho - \alpha$, where α is determined by the identity

$$\langle \mathfrak{l}, \alpha(Y)(\mathrm{d}f \otimes X) \rangle = \delta_1(f)\,(\partial\mathfrak{l})\,(X,Y)\,. \tag{67}$$

This corroborates that α is a bundle map from E to $\mathsf{T}M \otimes_M \mathsf{End}(E)$. □

Definition 8.5. Let $\mathrm{Der}_1(\mathsf{D}^\bullet(E), \pitchfork)$ be the space of degree 1 graded derivations ∂ of the reduced shuffle algebra that verify, for any $c \in \mathsf{D}^2(E)$ and any $X_i \in \mathsf{Sec}(E)$, $i = 1, 2, 3$,

$$\begin{aligned}(\partial c)(X_1, X_2, X_3) \;&=\; \sum_{i=1}^{3} (-1)^{i+1} \langle \partial(c(X_1, \ldots \hat{\imath} \ldots, X_3)), X_i \rangle \\ &\quad + \sum_{i<j} (-1)^i \, c(X_1, \ldots \hat{\imath} \ldots, \overset{(j)}{[X_i, X_j]_\partial}, \ldots, X_3)\,.\end{aligned} \tag{68}$$

A *homological vector field* of the supercommutative manifold $(M, \mathsf{D}^\bullet(E))$ is a square-zero derivation in $\mathrm{Der}_1(\mathsf{D}^\bullet(E), \pitchfork)$. Two homological vector fields of $(M, \mathsf{D}^\bullet(E))$ are *equivalent*, if they coincide on $C^\infty(M)$ and on $\mathsf{Sec}(E^*)$.

Observe that Equation (68) implies that two equivalent homological fields also coincide on $\mathsf{D}^2(E)$. We are now prepared to give the main theorem of this section.

Theorem 8.6. *Let E be a vector bundle. There exists a 1-to-1 correspondence between equivalence classes of homological vector fields*

$$\partial \in \mathrm{Der}_1(\mathsf{D}^\bullet(E), \pitchfork),\ \partial^2 = 0$$

and Loday algebroid structures on E.

Remark 8.7. This theorem is a kind of a non-antisymmetric counterpart of the well-known similar correspondence between homological vector fields of split supermanifolds and Lie algebroids. Furthermore, it may be viewed as an analogue for Loday algebroids of the celebrated Ginzburg-Kapranov correspondence for quadratic Koszul operads [7]. According to the latter result, homotopy Loday structures on a graded vector space V correspond bijectively to degree 1 differentials of the Zinbiel algebra $(\bar{\otimes} sV^*, \star)$, where s is the suspension operator and where $\bar{\otimes} sV^*$ denotes the reduced tensor module over sV^*. However, in our geometric setting scalars, or better functions, must be incorporated (see the proof of Theorem 8.6), which turns out to be impossible without passing from the Zinbiel multiplication or half shuffle $\star$ to its symmetrization $\pitchfork$. Moreover, it is clear that the algebraic structure on the function sheaf should be associative.

Proof. Let $([\cdot,\cdot], \rho, \alpha)$ be a Loday algebroid structure on the given vector bundle $E \to M$. According to Theorem 8.3, the corresponding coboundary operator ∂_B is a square 0 degree 1 graded derivation of the reduced shuffle algebra and (68) is satisfied by definition, as $[\cdot,\cdot]_{\partial_B} = [\cdot,\cdot]$.

Conversely, let ∂ be such a homological vector field. According to Theorem 8.4, the derivation ∂ coincides on $\mathsf{D}^0(E) \oplus \mathsf{D}^1(E)$ with ∂_B for a certain pseudoalgebra bracket $[\cdot,\cdot] = [\cdot,\cdot]_\partial$ on $\mathsf{Sec}(E)$. Its left anchor is $\rho = \rho_\partial$ and the generalized right anchor $b^r = \rho - \alpha$ is determined by means of formula (67), where $\mathfrak{l}$ runs through all sections of E^*.

To prove that the triplet $([\cdot,\cdot], \rho, \alpha)$ defines a Loday algebroid structure on E, it now suffices to check that the Jacobi identity holds true. It follows from (66) that

$$\langle \mathfrak{l}, [X_1, [X_2, X_3]] \rangle = -\langle \partial \langle \mathfrak{l}, X_1 \rangle, [X_2, X_3] \rangle + \langle \partial \langle \mathfrak{l}, [X_2, X_3] \rangle, X_1 \rangle - (\partial \mathfrak{l})(X_1, [X_2, X_3]).$$

Since the first term of the RHS is (up to sign) the evaluation of $[X_2, X_3]$ on the section $\partial\langle \mathfrak{l}, X_1 \rangle$ of E^*, and a similar remark is valid for the contraction $\langle \mathfrak{l}, [X_2, X_3] \rangle$ in the second term, we can apply (66) also to these two brackets. If we proceed analogously for $[[X_1, X_2], X_3]$ and

$[X_2,[X_1,X_3]]$, and use (65) and the homological property $\partial^2=0$, we find, after simplification, that the sum of the preceding three double brackets equals

$$\sum_{i=1}^{3}(-1)^{i+1}\rho(X_i)(\partial\mathfrak{l})(X_1,\ldots\hat{\imath}\ldots,X_3)+\sum_{i<j}(-1)^i\,(\partial\mathfrak{l})(X_1,\ldots\hat{\imath}\ldots,\overbrace{[X_i,X_j]}^{(j)},\ldots,X_3)\,.$$

In view of (68), the latter expression coincides with $(\partial^2\mathfrak{l})(X_1,X_2,X_3)=0$, so that the Jacobi identity holds.

It is clear that the just detailed assignment of a Loday algebroid structure to any homological vector field can be viewed as a map on equivalence classes of homological vector fields. □

Having a homological vector field ∂ associated with a Loday algebroid structure $([\cdot,\cdot],\rho,\alpha)$ on E, we can easily develop the corresponding Cartan calculus for the shuffle algebra $\mathcal{D}^\bullet(E)$.

Proposition 8.8. *For any $X\in\mathsf{Sec}(E)$, the contraction*

$$\mathcal{D}^p(E)\ni\ell\mapsto i_X\ell\in\mathcal{D}^{p-1}(E)\,,\quad (i_X\ell)(X_1,\ldots,X_{p-1})=\ell(X,X_1,\ldots,X_{p-1})\,,$$

is a degree -1 graded derivation of the shuffle algebra $(\mathcal{D}^\bullet(E),\pitchfork)$.

Proof. Using usual notations, our definitions, as well as a separation of the involved shuffles σ into the σ-s that verify $\sigma_1=1$ and those for which $\sigma_{p+1}=1$, we get

$$\left(i_{X_1}(\ell'\pitchfork\ell'')\right)(X_2,\ldots,X_{p+q})=\sum_{\sigma:\sigma_1=1}\operatorname{sign}\sigma\;(i_{X_1}\ell')(X_{\sigma_2},\ldots,X_{\sigma_p})\ell''(X_{\sigma_{p+1}},\ldots,X_{\sigma_{p+q}})$$

$$+\sum_{\sigma:\sigma_{p+1}=1}\operatorname{sign}\sigma\;\ell'(X_{\sigma_1},\ldots,X_{\sigma_p})(i_{X_1}\ell'')(X_{\sigma_{p+2}},\ldots,X_{\sigma_{p+q}}).$$

Whereas a (p,q)-shuffle of the type $\sigma_1=1$ is a $(p-1,q)$-shuffle with same signature, a (p,q)-shuffle such that $\sigma_{p+1}=1$ defines a $(p,q-1)$-shuffle with signature $(-1)^p\operatorname{sign}\sigma$. Therefore, we finally get

$$i_{X_1}(\ell'\pitchfork\ell'')=(i_{X_1}\ell')\pitchfork\ell''+(-1)^p\ell'\pitchfork(i_{X_1}\ell'').$$

□

Observe that the supercommutators $[i_X,i_Y]_{\mathrm{sc}}=i_Xi_Y+i_Yi_X$ do not necessarily vanish, so that the derivations i_X of the shuffle algebra generate a Lie superalgebra of derivations with negative degrees. Indeed, $[i_X,i_Y]_{\mathrm{sc}}=:i_{X\Box Y}$, $[[i_X,i_Y]_{\mathrm{sc}},i_Z]_{\mathrm{sc}}=:i_{(X\Box Y)\Box Z}$, ... are derivations of degree -2, -3, ... given on any $\ell\in\mathcal{D}^p(E)$ by

$$(i_{X\Box Y}\ell)(X_1,\ldots,X_{p-2})=\ell(Y,X,X_1,\ldots,X_{p-2})+\ell(X,Y,X_1,\ldots,X_{p-2})\,,$$

$$(i_{(X\Box X)\Box Y}\ell)(X_1,\ldots,X_{p-3})=2\ell(Y,X,X,X_1,\ldots,X_{p-3})-2\ell(X,X,Y,X_1,\ldots,X_{p-3})\,,\ldots$$

The next proposition is obvious.

Proposition 8.9. *The supercommutator $\mathcal{L}_X:=[\partial,i_X]_{\mathrm{sc}}=\partial i_X+i_X\partial$, $X\in\mathsf{Sec}(E)$, is a degree 0 graded derivation of the shuffle algebra. Explicitly, for any $\ell\in\mathcal{D}^p(E)$ and $X_1,\ldots,X_p\in\mathsf{Sec}(E)$,*

$$(\mathcal{L}_X\ell)(X_1,\ldots,X_p)=\rho(X)\left(\ell(X_1,\ldots,X_p)\right)-\sum_i\ell(X_1,\ldots,\overbrace{[X,X_i]}^{(i)},\ldots,X_p)\,.\qquad(69)$$

We refer to the derivation $\mathcal{L}_X$ as the Loday algebroid *Lie derivative along* X.

If we define the Lie derivative on the tensor algebra $T_{\mathbb{R}}(E) = \bigoplus_{p=0}^{\infty} \mathsf{Sec}(E)^{\otimes_{\mathbb{R}} p}$ in the obvious way by

$$\mathcal{L}_X(X_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} X_p) = \sum_i X_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} \overbrace{[X, X_i]}^{(i)} \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} X_p\,,$$

and if we use the canonical pairing

$$\langle \ell, X_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} X_p \rangle = \ell(X_1, \ldots, X_p)$$

between $\mathcal{D}^\bullet(E)$ and $T_{\mathbb{R}}(E)$, we get

$$\mathcal{L}_X \langle \ell, X_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} X_p \rangle = \langle \mathcal{L}_X \ell, X_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} X_p \rangle + \langle \ell, \mathcal{L}_X(X_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} X_p) \rangle\,. \tag{70}$$

The following theorem is analogous to the results in the standard case of a Lie algebroid $E = \mathsf{T}M$ and operations on the Grassmann algebra $\Omega(M) \subset \mathcal{D}^\bullet(\mathsf{T}M)$ of differential forms.

Theorem 8.10. *The graded derivations ∂, i_X, and $\mathcal{L}_X$ on $\mathcal{D}^\bullet(E)$ satisfy the following identities:*

(a) $2\partial^2 = [\partial, \partial]_{\mathrm{sc}} = 0$,

(b) $\mathcal{L}_X = [\partial, i_X]_{\mathrm{sc}} = \partial i_X + i_X \partial$,

(c) $\partial \mathcal{L}_X - \mathcal{L}_X \partial = [\partial, \mathcal{L}_X]_{\mathrm{sc}} = 0$,

(d) $\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = [\mathcal{L}_X, i_Y]_{\mathrm{sc}} = i_{[X,Y]}$,

(e) $\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = [\mathcal{L}_X, \mathcal{L}_Y]_{\mathrm{sc}} = \mathcal{L}_{[X,Y]}$.

Proof. The results (a), (b), and (c) are obvious. Identity (d) is immediately checked by direct computation. The last equality is a consequence of (c), (d), and the Jacobi identity applied to $[\mathcal{L}_X, [\partial, i_Y]_{\mathrm{sc}}]_{\mathrm{sc}}$. ◻

Note that we can easily calculate the Lie derivatives of negative degrees, $\mathcal{L}_{X \Box Y} := [\partial, i_{X \Box Y}]_{\mathrm{sc}}$, $\mathcal{L}_{(X \Box Y) \Box Z} := [\partial, i_{(X \Box Y) \Box Z}]_{\mathrm{sc}}$, ... with the help of the graded Jacobi identity.

Observe finally that Item (d) of the preceding theorem actually means that

$$i_{[X,Y]} = [\![i_X, i_Y]\!]_\partial,$$

where the RHS is the restriction to interior products of the derived bracket on $\mathrm{Der}(\mathcal{D}^\bullet(E), \pitchfork)$ defined by the graded Lie bracket $[\cdot,\cdot]_{\mathrm{sc}}$ and the interior Lie algebra derivation $[\partial, \cdot]_{\mathrm{sc}}$ of $\mathrm{Der}(\mathcal{D}^\bullet(E), \pitchfork)$ induced by the homological vector field ∂.

References

[1] D. Baraglia: Leibniz algebroids, twistings and exceptional generalized geometry, *arXiv:1101.0856.*

[2] Y. Bi and Y. Sheng: On higher analogues of Courant algebroid, *Sci. China Math.* **54** (2011), 437–447.

[3] T. J. Courant: Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 631–661.

[4] I. Y. Dorfman: Dirac structures of integrable evolution equations. *Phys. Lett. A* **125** (1987), 240–246.

[5] V. Drinfeld: Quantum groups, *Proceedings of the International Congress of Mathematicians* (Berkeley), American Mathematical Society, 1986.

[6] V. T. Filippov: n-Lie algebras, *Sibirsk. Math. Zh.* **26**(6) (1985), 126–140.

[7] V. Ginzburg, M. Kapranov: Koszul duality for operads, *Duke Math. J.*, **76(1)** (1994), 203–272.

[8] K. Grabowska, J. Grabowski: Variational calculus with constraints on general algebroids, *J. Phys. A: Math. Theor.* **41** (2008), 175204 (25pp).

[9] K. Grabowska, J. Grabowski and P. Urbański: Geometrical Mechanics on algebroids, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006), 559-575.

[10] J. Grabowski: Abstract Jacobi and Poisson structures. Quantization and star-products, *J. Geom. Phys.* **9** (1992), 45–73.

[11] J. Grabowski: Quasi-derivations and QD-algebroids, *Rep. Math. Phys.* **32** (2003), 445–451.

[12] J. Grabowski, M. De León, D. Martın de Diego, J. C. Marrero: Nonholonomic constraints: a new viewpoint, *J. Math. Phys.* **50** (2009), 013520 (17pp).

[13] J. Grabowski, G. Marmo: Non-antisymmetric versions of Nambu-Poisson and Lie algebroid brackets, *J. Phys. A: Math. Gen.* **34** (2001), 3803–3809.

[14] J. Grabowski, G. Marmo: Binary operations in classical and quantum mechanics, in *Classical and Quantum Integrability*, J. Grabowski and P. Urbański eds., *Banach Center Publ.* **59**, Warszawa 2003, 163–172.

[15] J. Grabowski and G. Marmo: The graded Jacobi algebras and (co)homology, *J. Phys. A: Math. Gen.* **36** (2003), 161–181.

[16] J. Grabowski, N. Poncin: Automorphisms of quantum and classical Poisson algebras, *Compositio Math.* **140** (2004), 511–527.

[17] J. Grabowski, P. Urbański: Algebroids – general differential calculi on vector bundles, *J. Geom. Phys.* **31** (1999), 111–141.

[18] J. C. Herz: Pseudo-algèbres de Lie, *C. R. Acad. Sci. Paris* **236** (1953), I, pp. 1935-1937, II, pp. 2289-2291.

[19] Y. Hagiwara: Nambu-Dirac manifolds, *J. Phys. A: Math. Gen.* **35** (2002), 1263-1281.

[20] Y. Hagiwara and T. Mizutani: Leibniz algebras associated with foliations, *Kodai Math. J.* **25** (2002), 151-165.

[21] R. Ibáñez, M. de León, J. C. Marrero, E. Padrón: Leibniz algebroid associated with a Nambu-Poisson structure, *J. Phys. A: Math. Gen.* **32** (1999), 8129–8144.

[22] N. Jacobson: On pseudo-linear transformations, *Proc. Nat. Acad. Sci.* **21** (1935), 667–670.

[23] N. Jacobson: Pseudo-linear transformations, *Ann. Math.* **38** (1937), 485–506.

[24] A. A. Kirillov: Local Lie algebras (Russian), *Uspekhi Mat. Nauk* **31** (1976), 57–76.

[25] Y. Kosmann-Schwarzbach: From Poisson algebras to Gerstenhaber algebras, *Ann. Inst. Fourier* **46** (1996), 1243–1274.

[26] Y. Kosmann-Schwarzbach and K. Mackenzie: Differential operators and actions of Lie algebroids, in *Quantization, Poisson brackets and beyond*, ed. T. Voronov, *Contemporary Math.* **315** (2002), 213-233.

[27] A. Kotov, T. Strobl: Generalizing geometryalgebroids and sigma models, in *Handbook of pseudo-Riemannian geometry and supersymmetry*, 209–262, *IRMA Lect. Math. Theor. Phys.* **16**, Eur. Math. Soc., Zrich, 2010.

[28] A. Lichnerowicz: Algèbre de Lie des automorphismes infinitésimaux d'une structure unimodulaire, *Ann. Inst. Fourier* **24** (1974), 219–266.

[29] J.-L. Loday: *Cyclic Homology*, Springer Verlag, Berlin 1992.

[30] J.-L. Loday: Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Ann. Inst. Fourier* **37** (1993), 269–93.

[31] J.-L. Loday and T. Pirashvili: Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Annalen* **296** (1993), 569–572.

[32] Zhang-Ju Liu, A. Weinstein, Ping Xu: Manin triples for Lie bialgebroids, *J. Diff. Geom.* **45** (1997), 547–574.

[33] K. C. H. Mackenzie: Lie algebroids and Lie pseudoalgebras, *Bull. London Math. Soc.* **27** (1995), 97-147.

[34] K. C. H. Mackenzie: General Theory of Lie Groupoids and Lie Algebroids, Cambridge University Press, 2005.

[35] K. C. H. Mackenzie and P. Xu: Lie bialgebroids and Poisson groupoids, *Duke Math. J.* **73** (1994), 415452.

[36] K. Mikami and T. Mizutani: Algebroids associated with pre-Poisson structures, *Noncommutative geometry and physics 2005*, 71-96, World Sci. Publ., Hackensack, NJ, 2007.

[37] E. Nelson: *Tensor Analysis*, Princeton University Press and The University of Tokyo Press, Princeton 1967.

[38] J. P. Ortega and V. Planas-Bielsa: Dynamics on Leibniz manifolds, *J. Geom. Phys.* **52**, (2004), 1-27.

[39] J. Peetre: Une caractérisation abstraite des opérateurs différentiels, *Math. Scand.* **7** (1959), 211–218.

[40] J. Peetre: Rectifications a l'article Une caractérisation abstraite des opératuers diffŕentiels, *Math. Scand.* **8** (1960), 116–120.

[41] J. Pradines: Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux, *C. R. Acad. Sci. Paris, Sér. A* **264** (1967), 245–248.

[42] D. Roytenberg: Courant algebroids, derived brackets and even symplectic supermanifolds, Ph. D. thesis, Berkeley, 1999.

[43] D. Roytenberg: On the structure of graded symplectic supermanifolds and Courant algebroids, In *Quantization, Poisson brackets and beyond (Manchester, 2001)*,*Contemp. Math.* **315**, Amer. Math. Soc., Providence, RI, 2002, pp. 169–185.

[44] P. Ševera and A. Weinstein: Poisson geometry with a 3-form background, *Prog. Theor. Phys. Suppl.* **144** (2001), 145–154.

[45] M. Stiénon, P. Xu: Modular classes of Loday algebroids, *C. R. Acad. Sci. Paris, Ser. I* **346** (2008), 193–198.

[46] K. Uchino: Remarks on the definition of a Courant algebroid, *Lett. Math. Phys.* **60** (2002), 171–175.

[47] A. M. Vinogradov: The logic algebra for the theory of linear differential operators, *Soviet. Mat. Dokl.* **13** (1972), 1058–1062.

[48] A. Wade: On some properties of Leibniz algebroids, *Infinite dimensional Lie groups in geometry and representation theory* (Washington, DC, 2000), 65-78, World Sci. Publ., River Edge, NJ, 2002.

Janusz GRABOWSKI
Polish Academy of Sciences
Institute of Mathematics
Śniadeckich 8, P.O. Box 21, 00-956 Warsaw, Poland
Email: jagrab@impan.pl

David KHUDAVERDYAN
University of Luxembourg
Campus Kirchberg, Mathematics Research Unit
6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg City, Grand-Duchy of Luxembourg
Email: david.khudaverdyan@uni.lu

Norbert PONCIN
University of Luxembourg
Campus Kirchberg, Mathematics Research Unit
6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg City, Grand-Duchy of Luxembourg
Email: norbert.poncin@uni.lu